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Novel Formulation for Flexible Multibody Dynamics

by

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Abstract

A novel approach is proposed for parallel computation in flexible multibody dynamics, based on a sub-domain decomposition technique. In this approach, the computational domain is divided into non-overlapping sub-domains and kinematic constraints are used to enforce the continuity of the displacement field over the entire structure. These kinematic constraints are enforced via fields of Lagrange multipliers that act at the interface between the sub-domains and can be interpreted as the interface connection forces. The proposed approach relies on a novel strategies for the enforcement of the kinematic constraints at the interface between sub-domains. The traditional approach **?** is to use global Lagrange multipliers to enforce all constraints. In the proposed approach, all constraints are enforced using local Lagrange multipliers and an interface mesh is defined as a byproduct. Furthermore, an augmented Lagrangian formulation is used in conjunction with with the local Lagrange multipliers. The penalty terms stemming from the augmented Lagrangian formulation provide a natural conditioning of the interface problem expressed in terms of the local Lagrange multipliers. In fact, as the penalty factor increases, the condition number of the interface problem flexibility matrix tends to unity. This advantage, however, comes at the expense of the solution of a large sized coarse mesh problem. To solve this latter problem, it is shown that the use of local Lagrange multipliers leads to an interface problem that can itself be decomposed into non-overlapping sub-domains. This contrasts with the traditional approaches for which this is not possible. Clearly, the proposed approach leads to a hierarchical decomposition of the problem, in which each decomposition leads to an new interface problem, of ever decreasing size. At the end, the overall problem can be solved without resorting to iterative solvers, achieving great computation efficiency and stability. Examples of application of the procedure will be presented for flexible multibody systems.

Contents

Chapter 1

Analysis of motion

1.1 General motion of a rigid body

Figure 1.1 depicts a rigid body defined in its reference configuration by frame $\mathcal{F}_0 = [\mathbf{A}, \mathcal{E}_0 = (\bar{e}_{01}, \bar{e}_{02}, \bar{e}_{03})]$. The position vector of point **A** with respect to point **O** is denoted $\underline{r}_0$. Let $\underline{r}_P$ be the position vector of a material point **P** of the rigid body with respect to inertial frame $\mathcal{F}^I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$. The position vector of the same material point with respect to point **A** is denoted $\underline{s}_P$. Hence, $\underline{r}_P = \underline{r}_0 + \underline{s}_P$.

The rigid body now undergoes an arbitrary motion that brings it to a final configuration defined by frame $\mathcal{F} = [\mathbf{A}, \mathcal{E} = (\bar{e}_1, \bar{e}_2, \bar{e}_3)]$. Let $\underline{\underline{R}}_0$ and $\underline{\underline{R}}$ be the rotation tensors that bring basis $\mathcal{I}$ to $\mathcal{E}_0$ and basis $\mathcal{E}_0$ to $\mathcal{E}$, respectively. Considering fig. 1.1, the following vector relationship is easily established,

$$\underline{u}_P = \underline{u} + \underline{S}_P - \underline{s}_P, \tag{1.1}$$

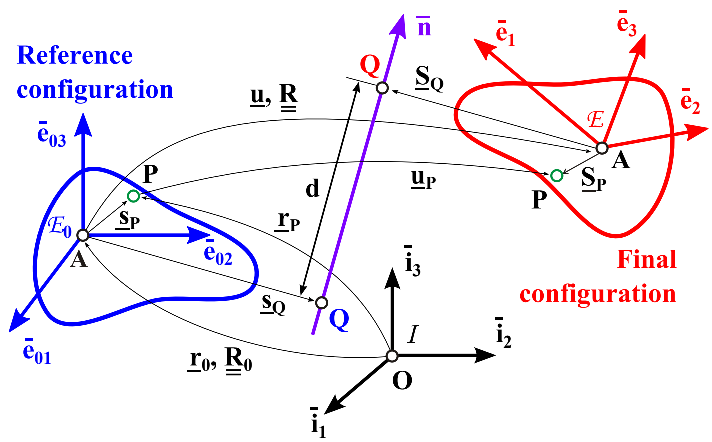

Figure 1.1: General motion of a rigid body.

where $\underline{S}_P$ is the position vector of material point **P** with respect to point **A** in the final configuration. Let $\underline{s}_P^* = \underline{\underline{R}}_0^T \underline{s}_P$ and $\underline{S}_P^+ = (\underline{\underline{R}}\,\underline{\underline{R}}_0)^T \underline{S}_P$ denote the components of vector $\underline{s}_P$ in basis $\mathcal{E}_0$ and of vector $\underline{S}_P$ in basis $\mathcal{E}$, respectively.

Because the body is assumed to be rigid, the components of vector $\underline{s}_P$ in $\mathcal{E}_0$ are identical to those of $\underline{S}_P$ in

$\mathcal{E}$, *i.e.*, $\underline{s}_P^* = \underline{S}_P^+$, and hence, $\underline{S}_P = \underline{\underline{R}}\,\underline{s}_P$. Equation (1.1) now becomes

$$\underline{u}_P = \underline{u} + \left(\underline{\underline{R}} - \underline{\underline{I}}\right)\underline{s}_P. \tag{1.2}$$

This relationship describes the displacement of a material point **P** of the rigid body in terms of $\underline{u}$, the displacement of its reference point, and tensor $\underline{\underline{R}}$ that defines its orientation. Note that the choice of reference point **A** is arbitrary, and hence, eq. (1.2) is not an intrinsic relationship.

To obtain a more general expression of the displacement field, the following question can be asked: is it possible to find a material point of the rigid body, say point **Q**, whose displacement is parallel to $\bar{n}$, the axis defining rotation tensor $\underline{\underline{R}}$? If point **Q** exist, its relative position vector, $\underline{s}_Q$, must satisfy the following relationship

$$\underline{u}_Q = \underline{u} + \left(\underline{\underline{R}} - \underline{\underline{I}}\right)\underline{s}_Q = d\bar{n}. \tag{1.3}$$

Constant d can be evaluated by taking the scalar product this equation by $\bar{n}^T$ to find $d = \bar{n}^T\underline{u}$. It then follows that

$$\left(\underline{\underline{R}} - \underline{\underline{I}}\right)\underline{s}_Q = d\bar{n} - \underline{u} = \left(\bar{n}\bar{n}^T - \underline{\underline{I}}\right)\underline{u}. \tag{1.4}$$

Using well-known identities, this equation can be written as $\widetilde{n}\left[2\sin\phi/2\ \underline{\underline{G}}\,\underline{s}_Q - \widetilde{n}\underline{u}\right] = 0$. The bracketed must be parallel to unit vector $\bar{n}$, which implies $2\sin\phi/2\ \underline{\underline{G}}\,\underline{s}_Q - \widetilde{n}\underline{u} = \beta\bar{n}$, where β is an arbitrary constant. The location of point **Q** is now readily found as

$$\underline{s}_Q = \frac{\widetilde{n}\underline{\underline{G}}^T}{2\sin\phi/2}\underline{u} + \frac{\beta}{2\sin\phi/2}\bar{n}.$$

This represents the equation of a line passing through point **Q** and parallel to $\bar{n}$. The displacements of all points on this line are along $\bar{n}$.

Point **Q** can be defined uniquely by requiring $\underline{s}_Q$ to be orthogonal to $\bar{n}$, *i.e.*, $\bar{n}^T\underline{s}_Q = 0$, and hence, $\beta = 0$. The location of point **Q** (Angeles, 1997) now becomes

$$\underline{s}_Q = \frac{\widetilde{n}\underline{\underline{G}}^T}{2\sin\phi/2}\underline{u}. \tag{1.5}$$

By construction, the displacement of point **Q** is parallel to $\bar{n}$, see eq. (1.3). Combining eqs. (1.2) and (1.3) now yields

$$\underline{u}_P = d\bar{n} + (\underline{\underline{R}} - \underline{\underline{I}})(\underline{s}_P - \underline{s}_Q). \tag{1.6}$$

This relationship expresses the displacement of a material point **P** of the rigid body as a translation, $d\bar{n}$, parallel to axis $\bar{n}$, followed by a rotation about that same axis. The displacement

$$d = \bar{n}^T \underline{u}, \tag{1.7}$$

is the *intrinsic displacement of the rigid body*: all points of the rigid body undergo the same displacement, d, followed by a rotation.

If the rigid body undergoes a general planar motion, $\underline{u}$ lies in the plane of the motion, and $\bar{n}$ is perpendicular this plane. Hence, $d = \bar{n}^T \underline{u} = 0$, the intrinsic displacement, d, of a rigid body in general planar motion always vanishes. If the rigid body undergoes a pure translation, axis $\bar{n}$ is along the displacement $\underline{u}$ of all the points of the body. The motion is then decomposed into a translation, $d\bar{n}$, followed by a rotation of vanishing magnitude about the same axis.

Equation (1.6) expresses the general motion of a rigid body as *screw motion* about axis $\bar{n}$. The *pitch of the screw*, ϖ, is defined as

$$\varpi = \frac{2\pi d}{\phi}. \tag{1.8}$$

Mozzi-Chasles' theorem due to Mozzi (1763) and Chasles (1830) states the results obtained here in a compact manner.

Theorem 1.1 (Mozzi-Chasles' theorem)**.** *The most general motion of a rigid body consists of a translation along an axis followed by a rotation about the same axis.*

The Mozzi-Chasles axis is defined by its orientation, $\bar{n}$, and the position of one of its points, $\underline{s}_Q$, given by eq. (1.5). Alternatively, this axis can be defined by its Plücker coordinates (Angeles, 1997, 1998)

$$\underline{\mathcal{Q}}_{MC} = \left\{ \begin{matrix} -\dfrac{\widetilde{n}\widetilde{n}\underline{\underline{G}}^T}{2\sin\phi/2}\underline{u} \\ \bar{n} \end{matrix} \right\} \tag{1.9}$$

1.1.1 Intrinsic representation of motion

In fig. 1.1, the general motion of a rigid body has been described by two quantities: the displacement of one of its material points and its rotation. Mathematically, these quantities are represented by vector $\underline{u}$, the displacement vector of material point **A**, and tensor $\underline{\underline{R}}$, which defines the rotation of the rigid body, respectively. Rotation tensor $\underline{\underline{R}}$ is an intrinsic quantity: it defines the rotation of the rigid body. In contrast, displacement vector $\underline{u}$ is not an intrinsic quantity: it represents the displacement of an arbitrarily chosen material point of the rigid body. Had another reference point been selected, say point **A'**, a different displacement vector, say $\underline{u}'$, would have resulted.

Mozzi-Chasles' theorem 1.1 states that a general rigid body motion consists of a translation along an axis followed by a rotation about the same axis. This theorem provides an alternative means of representing motion, which is fully defined by the magnitude of the translation, d, that of the rotation, ϕ, and the axis along and about which these motions are taking place.

This concept is illustrated in fig. 1.2, which depicts Mozzi-Chasles' axis and the magnitudes of the translation and rotation. Mozzi-Chasles' axis is a line, which is most effectively represented by its Plücker coordinates evaluated with respect to point **O**, $\underline{\mathcal{Q}}_{MC} = (\underline{w}, \bar{n})$, where $\bar{n}$ is the unit vector defining its orientation and $\underline{w} = \widetilde{r}_Q \bar{n}$. Vector $\underline{r}_Q$ is the position vector of an arbitrary point along Mozzi-Chasles' axis with respect to point **O**.

Mozzi-Chasles' theorem 1.1 now implies that a general motion, $\mathcal{M}$, is fully defined as follows

$$\mathcal{M} = (d, \phi, \underline{w}, \bar{n}). \tag{1.10}$$

The Plücker coordinates of Mozzi-Chasles' axis satisfy two constraints, $\|\bar{n}\| = 1$ and $\bar{n}^T \underline{w} = 0$. Consequently, motion $\mathcal{M}$ is characterized by six parameters only, as expected.

The representation of motion depicted in fig. 1.2 make no use of the configuration of a particular rigid body. The reference and final configurations of a specific rigid body undergoing the motion defined by eq. (1.10) are given in dotted lines. Note that the displacement of point **A**, an arbitrary material point of the rigid body, does not enter the definition of the motion defined by eq. (1.10), which now gives an *intrinsic* definition of the motion.

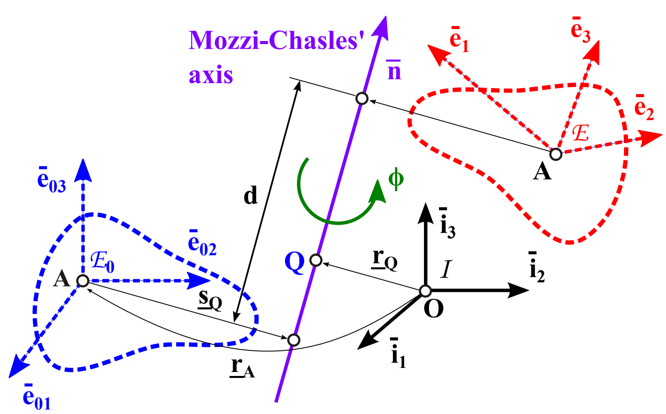

Figure 1.2: Definition of a general motion based on Mozzi-Chasles' theorem.

If the displacement of point **A** is desired, it can be obtained from eq. (1.6) as $\underline{u}_A = d\bar{n} + (\underline{\underline{R}} - \underline{\underline{I}})(-\underline{s}_Q)$. A cursory look at fig. 1.2 reveals that $\underline{r}_A + \underline{s}_Q + \beta\bar{n} = \underline{r}_Q$, and the displacement of point **A** becomes $\underline{u}_A = d\bar{n} + (\underline{\underline{R}} - \underline{\underline{I}})(\underline{r}_A - \underline{r}_Q + \beta\bar{n})$. Since $(\underline{\underline{R}} - \underline{\underline{I}})\bar{n} = \underline{0}$, this expression reduces to $\underline{u}_A = d\bar{n} + (\underline{\underline{R}} - \underline{\underline{I}})(\underline{r}_A - \underline{r}_Q)$, and finally,

$$\underline{u}_A = d\bar{n} + 2\sin\frac{\phi}{2}\underline{\underline{G}}\,\underline{w} + (\underline{\underline{R}} - \underline{\underline{I}})\underline{r}_A. \tag{1.11}$$

The first two terms of this equation represent the intrinsic contribution to the displacement of any arbitrary point. The third term represents the additional contribution that is specific to point **A**; as expected, it depends on the position vector, $\underline{r}_A$, of point **A**.

Example 1.1. ***Displacement of the points located on a circular cylinder***

Consider a general motion, $\mathcal{M} = (d, \phi, \underline{Q}_{MC})$, and a circular cylinder of radius ϱ whose axis coincides with Mozzi-Chasles' axis. Find the magnitude of the displacement of the points located on this cylinder.

The displacement vector of an arbitrary point **A** resulting from motion $\mathcal{M}$ is given by eq. (1.6) as $\underline{u}_A = d\bar{n} - (\underline{\underline{R}} - \underline{\underline{I}})\underline{s}_Q = d\bar{n} - 2\sin\phi/2\ \widetilde{n}\underline{s}_Q$. For all points **A** located on a circular cylinder of radius ϱ whose axis coincides with Mozzi-Chasles' axis, $\underline{s}_Q = \varrho\bar{u}$, where $\bar{n}^T\bar{u} = 0$. It follows that $\underline{u}_A = d\bar{n} - 2\varrho\sin\phi/2\ \bar{v}$, where $\bar{n}^T\bar{v} = 0$. The square of the norm of the displacement of point **A** now becomes $\|\underline{u}_A\|^2 = d^2 + (2\varrho\sin\phi/2)^2$. Because d and ϕ are constants for the given motion, $\mathcal{M}$, and ϱ is the radius of the circular cylinder whose axis coincides with Mozzi-Chasles' axis, it follows that the magnitudes of the displacement of all points located on this circular are identical and given by the above formula.

1.2 The motion tensor

In this section, the motion tensor is introduced as the tensor that relates the Plücker coordinates of a line of a rigid body in its initial and final configurations.

1.2.1 Transformation of a line of a rigid body

Figure 1.3 shows a rigid body in its reference configuration defined by frame $\mathcal{F}^I = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_1, \bar{\imath}_2, \bar{\imath}_3)]$. Two points of this rigid body, denoted points **P** and **Q**, are defined by their position vectors with respect to point **O** given as $\underline{s}_P$ and $\underline{s}_Q$, respectively. In the final configuration, the rigid body is associated with frame $\mathcal{F} = [\mathbf{A}, \mathcal{B}^* = (\bar{b}_1, \bar{b}_2, \bar{b}_3)]$. Superscripts $(\cdot)^*$ indicate tensor components resolved in basis $\mathcal{B}^*$. The position vectors of material points **P** and **Q** with respect to point **A** are now $\underline{S}_P$ and $\underline{S}_Q$, respectively. Because points **P** and **Q** are material points of the rigid body, $\underline{S}_P = \underline{\underline{R}}\,\underline{S}^*_P$ and $\underline{S}_Q = \underline{\underline{R}}\,\underline{S}^*_Q$.

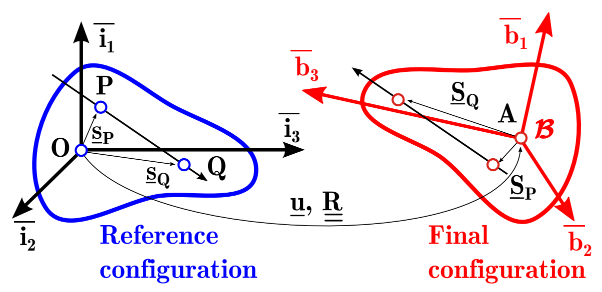

Figure 1.3: A line of a rigid body in the reference and final configurations.

Consider now the line passing through these two points in the final configuration. Its orientation, resolved in basis $\mathcal{B}^*$, is $\bar{\ell}^* = (\underline{S}^*_Q - \underline{S}^*_P)/(\|\underline{S}^*_Q - \underline{S}^*_P\|)$. The Plücker coordinates of this line evaluated with respect to point $\mathbf{A}$, are

$$\underline{\mathcal{Q}}^* = \begin{Bmatrix} \widetilde{S}^*_P \bar{\ell}^* \\ \bar{\ell}^* \end{Bmatrix} = \begin{Bmatrix} \underline{k}^* \\ \bar{\ell}^* \end{Bmatrix}. \tag{1.12}$$

The Plücker coordinates of the same line with respect to point $\mathbf{O}$ will now be evaluated and resolved in basis $\mathcal{I}$. First, the orientation of the line is now

$$\bar{\ell} = \frac{(\underline{u} + \underline{S}_Q) - (\underline{u} + \underline{S}_P)}{\|(\underline{u} + \underline{S}_Q) - (\underline{u} + \underline{S}_P)\|} = \frac{\underline{S}_Q - \underline{S}_P}{\|\underline{S}_Q - \underline{S}_P\|} = \underline{\underline{R}}\,\frac{\underline{S}^*_Q - \underline{S}^*_P}{\|\underline{S}^*_Q - \underline{S}^*_P\|} = \underline{\underline{R}}\,\bar{\ell}^*.$$

Next, the Plücker coordinates of the same line become

$$\underline{\mathcal{Q}} = \begin{Bmatrix} (\widetilde{u} + \widetilde{S}_P)\bar{\ell} \\ \bar{\ell} \end{Bmatrix} = \begin{Bmatrix} \widetilde{u}\underline{\underline{R}}\bar{\ell}^* + \underline{\underline{R}}\widetilde{S}^*_P \underline{\underline{R}}^T \underline{\underline{R}}\,\bar{\ell}^* \\ \underline{\underline{R}}\,\bar{\ell}^* \end{Bmatrix} = \begin{bmatrix} \underline{\underline{R}} & \widetilde{u}\underline{\underline{R}} \\ 0 & \underline{\underline{R}} \end{bmatrix} \begin{Bmatrix} \widetilde{S}^*_P \bar{\ell}^* \\ \bar{\ell}^* \end{Bmatrix}. \tag{1.13}$$

The *motion tensor* is defined as

$$\underline{\underline{\mathcal{C}}} = \begin{bmatrix} \underline{\underline{R}} & \widetilde{u}\underline{\underline{R}} \\ \underline{\underline{0}} & \underline{\underline{R}} \end{bmatrix}, \tag{1.14}$$

and eq. (1.13) can now be written in a compact form as

$$\underline{\mathcal{Q}} = \begin{Bmatrix} \underline{k} \\ \bar{\ell} \end{Bmatrix} = \underline{\underline{\mathcal{C}}}\,\underline{\mathcal{Q}}^* = \underline{\underline{\mathcal{C}}} \begin{Bmatrix} \underline{k}^* \\ \bar{\ell}^* \end{Bmatrix}. \tag{1.15}$$

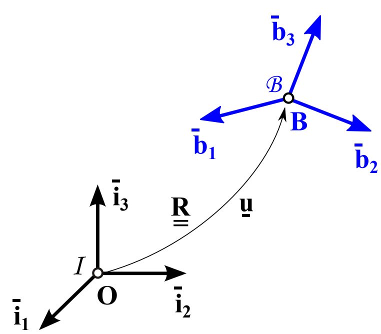

Figure 1.4: Two frames with a relative displacement, $\underline{u}$, and a relative rotation, $\underline{\underline{R}}$.

Clearly, the motion tensor relates the Plücker coordinates of an arbitrary line of the rigid body resolved in two frames. This change of frame operation is more complex than the change in basis operation: it involve both a change of basis and a change of reference point (Bottema and Roth, 1979; Pradeep et al., 1989; Angeles, 1998). Equation (1.15) can be written in a more explicit manner as $\underline{\mathcal{Q}}^{[\mathcal{F}^I]} = \underline{\underline{\mathcal{C}}}^{[\mathcal{F}^I]}\underline{\mathcal{Q}}^{[\mathcal{F}]}$. In this form, the present change of frame operation mirrors the change of basis operation.

Figure 1.4 depicts the change of frame operation in a more abstract manner: the motion brings frame $\mathcal{F}^{\mathcal{I}} = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_i, \bar{\imath}_2, \bar{\imath}_3)]$ to frame $\mathcal{F} = [\mathbf{B}, \mathcal{B} = (\bar{b}_1, \bar{b}_2\bar{b}_3)]$. Vector $\underline{u}$ is the relative position of point $\mathbf{B}$ with respect to point $\mathbf{O}$ and rotation tensor $\underline{\underline{R}}$ brings basis $\mathcal{I}$ to basis $\mathcal{B}$.

Let $\underline{\mathcal{I}}_i$ denote a null vector with a single unit entry in location i. Application of the motion tensor to these unit vectors yields

$$\underline{\mathcal{B}}_i = \underline{\underline{\mathcal{C}}}\,\underline{\mathcal{I}}_i. \tag{1.16}$$

In view of the definition of the motion tensor, eq. (1.14), the following results are found easily

$$\underline{\mathcal{B}}_1 = \left\{ \begin{matrix} \bar{b}_1 \\ \underline{0} \end{matrix} \right\}, \quad \underline{\mathcal{B}}_2 = \left\{ \begin{matrix} \bar{b}_2 \\ \underline{0} \end{matrix} \right\}, \quad \underline{\mathcal{B}}_3 = \left\{ \begin{matrix} \bar{b}_3 \\ \underline{0} \end{matrix} \right\}, \tag{1.17}$$

because the rotation tensor can be expressed as $\underline{\underline{R}} = \left[\bar{b}_1, \bar{b}_2, \bar{b}_3 \right]$. On the other hand, the last three vectors are

$$\underline{\mathcal{B}}_4 = \left\{ \begin{matrix} \widetilde{u}\bar{b}_1 \\ \bar{b}_1 \end{matrix} \right\}, \quad \underline{\mathcal{B}}_5 = \left\{ \begin{matrix} \widetilde{u}\bar{b}_2 \\ \bar{b}_2 \end{matrix} \right\}, \quad \underline{\mathcal{B}}_6 = \left\{ \begin{matrix} \widetilde{u}\bar{b}_3 \\ \bar{b}_3 \end{matrix} \right\}. \tag{1.18}$$

Note that vectors $\underline{\mathcal{I}}_i$, $i = 4, 5, 6$ can be interpreted as the Plücker coordinates of lines $\mathcal{L}_1 = (\mathbf{O}, \bar{\imath}_1)$, $\mathcal{L}_2 = (\mathbf{O}, \bar{\imath}_2)$, and $\mathcal{L}_3 = (\mathbf{O}, \bar{\imath}_3)$, respectively, evaluated with respect to point $\mathbf{O}$. Notation $\mathcal{L} = (\mathbf{P}, \bar{\ell})$ indicates a line passing through point $\mathbf{P}$ and of orientation defined by unit vector $\bar{\ell}$. Equation (1.15) then implies that vectors $\underline{\mathcal{B}}_4$, $\underline{\mathcal{B}}_5$, and $\underline{\mathcal{B}}_6$ can be interpreted as the Plücker coordinates of lines $\mathcal{L}_4 = (\mathbf{B}, \bar{b}_1)$, $\mathcal{L}_5 = (\mathbf{B}, \bar{b}_2)$, and $\mathcal{L}_6 = (\mathbf{B}, \bar{b}_3)$, evaluated with respect to point $\mathbf{O}$.

1.2.2 Properties of the motion tensor

The motion tensor can be factorized in the following manner

$$\underline{\underline{\mathcal{C}}} = \begin{bmatrix} \underline{\underline{I}} & \widetilde{u} \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix} \begin{bmatrix} \underline{\underline{R}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{R}} \end{bmatrix} = \underline{\underline{\mathcal{T}}}\,\underline{\underline{\mathcal{R}}}, \tag{1.19}$$

where $\underline{\underline{\mathcal{R}}}$ is the *rotation tensor* and $\underline{\underline{\mathcal{T}}}$ the *translation tensor*. The eigenvalues of the motion tensor are easily obtained from its characteristic equation, $\det(\underline{\underline{\mathcal{C}}} - \lambda \underline{\underline{\mathcal{I}}}) = 0$. Given the structure of the motion tensor given by eq. (1.14), the characteristic equation reduces to $\det^2(\underline{\underline{R}} - \lambda \underline{\underline{I}}) = 0$, which implies that the eigenvalues of the motion tensor are identical to those of the rotation tensor, but each with a multiplicity of two. The motion tensor, however, unlike the rotation tensor, is not an orthogonal tensor.

The inverse of the motion tensor is found easily as

$$\underline{\underline{\mathcal{C}}}^{-1} = \underline{\underline{\mathcal{R}}}^{-1}\underline{\underline{\mathcal{T}}}^{-1} = \underline{\underline{\mathcal{R}}}^{T}\underline{\underline{\mathcal{T}}}^{-1} = \begin{bmatrix} \underline{\underline{R}}^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{R}}^T \end{bmatrix} \begin{bmatrix} \underline{\underline{I}} & \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{I}} \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}^T & \underline{\underline{R}}^T\widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{R}}^T \end{bmatrix}. \tag{1.20}$$

Two linearly independent eigenvectors of the motion tensor associated with its unit eigenvalues are found to be

$$\underline{\mathcal{N}}_1^{\dagger} = \begin{Bmatrix} \bar{n} \\ \underline{0} \end{Bmatrix}, \quad \text{and} \quad \underline{\mathcal{N}}_2^{\dagger} = \begin{Bmatrix} \dfrac{\underline{\underline{G}}^T\underline{u}}{2\sin\phi/2} \\ \bar{n} \end{Bmatrix}. \tag{1.21}$$

The fact that $\underline{\mathcal{N}}_1^{\dagger}$ is an eigenvector of the motion tensor stems from the corresponding property for the rotation tensor, $\underline{\underline{R}}\,\bar{n} = \bar{n}$. It is readily verified that $\underline{\mathcal{N}}_2^{\dagger}$ is also an eigenvector of the motion tensor, indeed, $\underline{\underline{R}}\,\underline{\underline{G}}^T\underline{u}/(2\sin\phi/2) + \widetilde{u}\underline{\underline{R}}\,\bar{n} = (\underline{\underline{G}} - 2\widetilde{n}\sin\phi/2)\underline{u}/(2\sin\phi/2) = \underline{\underline{G}}^T\underline{u}/(2\sin\phi/2)$.

Any linear combination of eigenvectors $\underline{\mathcal{N}}_1^{\dagger}$ and $\underline{\mathcal{N}}_2^{\dagger}$ is still an eigenvector of the motion tensor. Consequently, the family of eigenvectors associated with the unit eigenvalue is expressed as follows

$$\underline{\mathcal{N}} = \begin{Bmatrix} \underline{m} \\ \bar{n} \end{Bmatrix} = \frac{(\alpha-1)d}{2\sin\phi/2}\underline{\mathcal{N}}_1^{\dagger} + \underline{\mathcal{N}}_2^{\dagger}, \tag{1.22}$$

where α is an arbitrary scalar and d the intrinsic displacement of the rigid body. The displacement related part of the eigenvector is

$$\underline{m} = \frac{\underline{\underline{G}}^T\underline{u}}{2\sin\phi/2} + \frac{(\alpha-1)d}{2\sin\phi/2}\bar{n}. \tag{1.23}$$

The scalar product of the two vectors forming the eigenvector is closely related to the intrinsic displacement of the rigid body

$$\lambda = \bar{n}^T\underline{m} = \frac{\alpha d}{2\sin\phi/2}. \tag{1.24}$$

1.2.3 Mozzi-Chasles' axis

In general, an arbitrary line of a rigid body is different in the reference and final configurations. The following question can then be asked: is it possible to find a line of the rigid body that is identical in the reference and final configurations? If such line exists, its Plücker coordinates in the reference and final configurations are identical, *i.e.*, $\underline{\mathcal{Q}} = \underline{\mathcal{Q}}^*$, or, using eq. (1.15), $\underline{\mathcal{Q}} = \underline{\underline{\mathcal{C}}}\,\underline{\mathcal{Q}}$.

This implies that the Plücker coordinates of this line must form an eigenvector of the motion tensor, as given by eq. (1.22). Because the first three components of the Plücker coordinates of a line must be orthogonal to the last three, eq. (1.24) implies $\lambda = \alpha = 0$, and hence,

$$\underline{\mathcal{Q}}_{MC} = \underline{\mathcal{N}}_2^\dagger - \frac{d}{2\sin\phi/2}\underline{\mathcal{N}}_1^\dagger = \left\{ \begin{matrix} -\dfrac{\underline{\underline{G}}^T \widetilde{n}\widetilde{n}}{2\sin\phi/2}\underline{u} \\ \bar{n} \end{matrix} \right\}. \tag{1.25}$$

In summary, the Plücker coordinates of the line of the rigid body that is identical in the reference and final configurations are given by eq. (1.25). These coordinates are those of Mozzi-Chasles' axis. Hence, Mozzi-Chasles' axis is the line of the rigid body that is identical in the reference and final configurations. This can be written as $\underline{\mathcal{Q}}_{MC} = \underline{\underline{\mathcal{C}}}\,\underline{\mathcal{Q}}_{MC}$: Mozzi-Chasles' axis is an eigenvector of the motion tensor corresponding to a unit eigenvalue.

1.2.4 Intrinsic expression of the motion tensor

The motion tensor was defined by eq. (1.14), which is not an intrinsic expression because the displacement vector of the reference point of the rigid body, $\underline{u}$, explicitly appears in this definition. In this section, an intrinsic expression of the motion tensor is sought, *i.e.*, an expression in which vector $\underline{u}$ does not appear explicitly.

Rodrigues' rotation formula provides an intrinsic equation for the rotation tensor in terms of $\bar{n}$, the eigenvector of the rotation tensor associated with its unit eigenvalue, and ϕ, the magnitude of the rotation. A similar approach is followed here for the motion tensor, which should be expressed in terms of $\underline{\mathcal{N}}$, the eigenvector of the motion tensor associated with its unit eigenvalue, ϕ, the magnitude of the rotation, and d, the intrinsic displacement of the rigid body.

The motion tensor, eq. (1.14), is composed of two sub-matrices: the rotation tensor, repeated twice along the diagonal, and tensor $\widetilde{u}\underline{\underline{R}}$, appearing as an off-diagonal term. The intrinsic expression of the rotation tensor is provided by Rodrigues' rotation formula. In contrast, the term $\widetilde{u}\underline{\underline{R}}$ is not intrinsic because the displacement vector of the reference point, $\underline{u}$, appear explicitly.

Using the definition of the intrinsic displacement of the rigid body, the displacement vector is related to the eigenvector of the motion tensor, with the help of eq. (1.23) to find $\underline{m} = \left[\underline{\underline{G}}^T\underline{u} + (\alpha - 1)\bar{n}\bar{n}^T\underline{u}\right]/(2\sin\phi/2)$. Introducing the expression for the half-angle rotation tensor then yields

$$\underline{m} = \underline{\underline{E}}\,\underline{u}, \tag{1.26}$$

where second-order tensor $\underline{\underline{E}}$ is defined as

$$\underline{\underline{E}} = \frac{\alpha}{2\sin\phi/2}\underline{\underline{I}} - \frac{1}{2}\widetilde{n} + \left(\frac{\alpha}{2\sin\phi/2} - \frac{1}{2\tan\phi/2}\right)\widetilde{n}\widetilde{n}. \tag{1.27}$$

It now becomes possible to express the displacement vector in terms of the first part of the eigenvector of the motion tensor as

$$\underline{u} = \underline{\underline{J}}\,\underline{m}, \tag{1.28}$$

where tensor $\underline{\underline{J}} = \underline{\underline{E}}^{-1}$ is easily found as

$$\underline{\underline{J}} = \frac{2\sin\phi/2}{\alpha}\underline{\underline{I}} + (1-\cos\phi)\widetilde{n} + \left(\frac{2\sin\phi/2}{\alpha} - \sin\phi\right)\widetilde{n}\widetilde{n}. \tag{1.29}$$

Equation (1.28) now yields an explicit expression of the displacement of the body's reference point

$$\widetilde{u} = \widetilde{\underline{\underline{J}}\,\underline{m}} = \sin\phi\ \widetilde{m} + d(1-\alpha\cos\frac{\phi}{2})\widetilde{n} + (1-\cos\phi)(\widetilde{n}\widetilde{m} - \widetilde{m}\widetilde{n}). \tag{1.30}$$

Finally, tedious algebra reveals the following result,

$$\widetilde{u}\underline{\underline{R}} = \widetilde{\underline{\underline{J}}\,\underline{m}}\,\underline{\underline{R}} = \sin\phi\,\widetilde{m} + dc_1\widetilde{n} + (1-\cos\phi)\left(\widetilde{n}\widetilde{m} + \widetilde{m}\widetilde{n}\right) + dc_2\widetilde{n}\widetilde{n}, \tag{1.31}$$

where coefficients c_1 and c_2 are defined as

$$c_1 = \cos\phi - \quad \alpha\cos\phi/2, \tag{1.32a}$$

$$c_2 = \sin\phi - 2\alpha\sin\phi/2. \tag{1.32b}$$

Combining Rodrigues' rotation formula and eq. (1.31), the motion tensor, eq. (1.14), becomes

$$\underline{\underline{\mathcal{C}}} = \underline{\underline{\mathcal{I}}} + \begin{bmatrix} \sin\phi\,\underline{\underline{I}} & dc_1\underline{\underline{I}} \\ \underline{\underline{0}} & \sin\phi\,\underline{\underline{I}} \end{bmatrix} \begin{bmatrix} \widetilde{n} & \widetilde{m} \\ \underline{\underline{0}} & \widetilde{n} \end{bmatrix} + \begin{bmatrix} (1-\cos\phi)\,\underline{\underline{I}} & dc_2\underline{\underline{I}} \\ \underline{\underline{0}} & (1-\cos\phi)\,\underline{\underline{I}} \end{bmatrix} \begin{bmatrix} \widetilde{n} & \widetilde{m} \\ \underline{\underline{0}} & \widetilde{n} \end{bmatrix} \begin{bmatrix} \widetilde{n} & \widetilde{m} \\ \underline{\underline{0}} & \widetilde{n} \end{bmatrix}. \tag{1.33}$$

To simplify the writing of this seemingly complicated expression, the following notation is introduced. First, tensor $\underline{\underline{\mathcal{Z}}}$, a function of two scalars, α and β, is introduced

$$\underline{\underline{\mathcal{Z}}}(\alpha, \beta) = \begin{bmatrix} \beta \underline{\underline{I}} & \alpha \underline{\underline{I}} \\ \underline{\underline{0}} & \beta \underline{\underline{I}} \end{bmatrix}. \tag{1.34}$$

Second, the *generalized vector product tensor* is defined

$$\widetilde{\mathcal{N}} = \begin{bmatrix} \widetilde{n} & \widetilde{m} \\ \underline{\underline{0}} & \widetilde{n} \end{bmatrix}. \tag{1.35}$$

Notation $\widetilde{\mathcal{N}}$ does not indicate a 6×6 skew-symmetric tensor, but rather the above 6×6 tensor formed by three skew-symmetric sub-tensors.

Introducing these various notations into eq. (1.33) yields the desired intrinsic expression of the motion tensor and of its inverse

$$\underline{\underline{\mathcal{C}}}(\underline{\mathcal{N}}) = \underline{\underline{\mathcal{I}}} + \underline{\underline{\mathcal{Z}}}(dc_1, \sin\phi)\widetilde{\mathcal{N}} + \underline{\underline{\mathcal{Z}}}(dc_2, 1 - \cos\phi)\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}, \tag{1.36a}$$

$$\underline{\underline{\mathcal{C}}}^{-1}(\underline{\mathcal{N}}) = \underline{\underline{\mathcal{I}}} - \underline{\underline{\mathcal{Z}}}(dc_1, \sin\phi)\widetilde{\mathcal{N}} + \underline{\underline{\mathcal{Z}}}(dc_2, 1 - \cos\phi)\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}. \tag{1.36b}$$

The parallel between this intrinsic expression for the motion tensor and that for the rotation tensor given by Rodrigues' rotation formula, is striking. Clearly, the skew-symmetric tensor, $\widetilde{n}$, appearing in the expression for the rotation tensor is replaced by the generalized vector product tensor, $\widetilde{\mathcal{N}}$, appearing in that for the motion tensor. The two scalars, $\sin\phi$ and $(1 - \cos\phi)$, appearing in the expression for the rotation tensor becomes the second arguments of tensor $\underline{\underline{\mathcal{Z}}}$ appearing in that for the motion tensor.

Rodrigues' rotation formula provides an intrinsic expression for the rotation tensor and is a direct consequence of Euler's theorem on rotations. Similarly, the intrinsic expression for the motion tensor is a direct consequence of the Mozzi-Chasles theorem.

1.2.5 Properties of the generalized vector product tensor

The generalized vector product tensor defined by eq. (1.35) enjoys remarkable properties that generalize those of the skew-symmetric tensor. First, the skew-symmetric operator, $\widetilde{n}$, possesses a null eigenvalue, $\widetilde{n}\bar{n} = 0\bar{n}$. Similarly, the generalized vector product tensor also possesses a null eigenvalue, $\widetilde{\mathcal{N}}\underline{\mathcal{N}} = 0\underline{\mathcal{N}}$.

The second property of the generalized vector product tensor generalizes the behavior of the skew-symmetric tensor under a change of basis operation. Consider the following triple matrix product

$$\begin{bmatrix} \widetilde{n}_3 & \widetilde{m}_3 \\ \underline{\underline{0}} & \widetilde{n}_3 \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}_2^T & \underline{\underline{R}}_2^T \widetilde{u}_2^T \\ \underline{\underline{0}} & \underline{\underline{R}}_2^T \end{bmatrix} \begin{bmatrix} \widetilde{n}_1 & \widetilde{m}_1 \\ \underline{\underline{0}} & \widetilde{n}_1 \end{bmatrix} \begin{bmatrix} \underline{\underline{R}}_2 & \widetilde{u}_2 \underline{\underline{R}}_2 \\ \underline{\underline{0}} & \underline{\underline{R}}_2 \end{bmatrix}.$$

This equality implies two conditions. The first condition is $\widetilde{n}_3 = \underline{\underline{R}}_2^T \widetilde{n}_1 \underline{\underline{R}}_2$, which implies $\bar{n}_3 = \underline{\underline{R}}_2^T \bar{n}_1$. The second condition is $\widetilde{m}_3 = \underline{\underline{R}}_2^T (\widetilde{m}_1 + \widetilde{n}_1 \widetilde{u}_2 - \widetilde{u}_2 \widetilde{n}_1) \underline{\underline{R}}_2$, and tensor identities then lead to $\underline{m}_3 = \underline{\underline{R}}_2^T (\underline{m}_1 + \widetilde{n}_1 \underline{u}_2)$. These results can be summarized by the following equivalence,

$$\widetilde{\mathcal{N}}_3 = \underline{\underline{\mathcal{C}}}^{-1}(\underline{\mathcal{N}}_2) \widetilde{\mathcal{N}}_1 \underline{\underline{\mathcal{C}}}(\underline{\mathcal{N}}_2) \Longleftrightarrow \underline{\mathcal{N}}_3 = \underline{\underline{\mathcal{C}}}^{-1}(\underline{\mathcal{N}}_2) \underline{\mathcal{N}}_1. \tag{1.37}$$

The third property of the generalized vector product tensor generalizes identity, which holds for unit vectors and is rewritten here as $\widetilde{n}\widetilde{n}\widetilde{n} + \widetilde{n} = 0$.

$$\widetilde{\mathcal{N}}\widetilde{\mathcal{N}}\widetilde{\mathcal{N}} + \underline{\underline{\mathcal{Z}}}(2\lambda, 1)\widetilde{\mathcal{N}} = \underline{\underline{0}}. \tag{1.38}$$

The use of well-known identities yields the above result, where $\lambda = \bar{n}^T \underline{m}$.

Consider two vectors defined as

$$\underline{\mathcal{V}} = \begin{Bmatrix} \underline{v} \\ \underline{\omega} \end{Bmatrix}, \quad \underline{\mathcal{P}} = \begin{Bmatrix} \underline{p} \\ \underline{q} \end{Bmatrix}.$$

The well-known property of the vector product, $\widetilde{a}\underline{b} = -\widetilde{b}\underline{a}$, then generalizes to

$$\widetilde{\mathcal{V}}\underline{\mathcal{P}} = -\widetilde{\mathcal{P}}\underline{\mathcal{V}}. \tag{1.39}$$

The following operation is alos needed

$$\widetilde{\mathcal{V}}^T \underline{\mathcal{P}} = \widehat{\mathcal{P}}\underline{\mathcal{V}}, \tag{1.40}$$

where the following notation was introduced

$$\widehat{\mathcal{P}} = \begin{bmatrix} \underline{\underline{0}} & \widetilde{p} \\ \widetilde{p} & \widetilde{q} \end{bmatrix}. \tag{1.41}$$

1.2.6 Composition of motion

Figure 1.5 shows three frames denoted $\mathcal{F}^{\mathcal{I}} = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_i, \bar{\imath}_2, \bar{\imath}_3)]$, $\mathcal{F}^k = [\mathbf{K}, \mathcal{B}^k = (\bar{b}_1^k, \bar{b}_2^k \bar{b}_3^k)]$, and $\mathcal{F}^\ell = [\mathbf{L}, \mathcal{B}^\ell = (\bar{b}_1^\ell, \bar{b}_2^\ell \bar{b}_3^\ell)]$. The relative position vectors of points $\mathbf{K}$ and $\mathbf{L}$ with respect to point $\mathbf{O}$ are denoted $\underline{u}^k$ and $\underline{u}^\ell$, respectively. The relative rotation tensors of bases $\mathcal{B}^k$ and $\mathcal{B}^\ell$ with respect to basis $\mathcal{I}$ are denoted $\underline{\underline{R}}^k$ and $\underline{\underline{R}}^\ell$, respectively. Finally, let $\underline{\underline{\mathcal{C}}}^k$, $\underline{\underline{\mathcal{C}}}^\ell$, and $\underline{\underline{\mathcal{C}}}$ denote the motion tensors that bring frame $\mathcal{F}^\mathcal{I}$ to $\mathcal{F}^k$, frame $\mathcal{F}^\mathcal{I}$ to $\mathcal{F}^\ell$, and frame $\mathcal{F}^k$ to $\mathcal{F}^\ell$, respectively. All tensor components are resolved in frame $\mathcal{F}^\mathcal{I}$.

In view of eq. (1.16), $\underline{\mathcal{B}}_i^k = \underline{\underline{\mathcal{C}}}^k \underline{\mathcal{J}}_i$ and $\underline{\mathcal{B}}_i^\ell = \underline{\underline{\mathcal{C}}}^\ell \underline{\mathcal{J}}_i$, leading to $\underline{\mathcal{B}}_i^\ell = \underline{\underline{\mathcal{C}}}^\ell \underline{\underline{\mathcal{C}}}^{k-1} \underline{\mathcal{B}}_i^k$. Because $\underline{\mathcal{B}}_i^\ell = \underline{\underline{\mathcal{C}}}\, \underline{\mathcal{B}}_i^k$, it then follows that

$$\underline{\underline{\mathcal{C}}} = \underline{\underline{\mathcal{C}}}^\ell \underline{\underline{\mathcal{C}}}^{k-1}. \tag{1.42}$$

This tensor relationship is called *composition of motion*: it expresses the relative motion tensor, $\underline{\underline{\mathcal{C}}}$, of frame $\mathcal{F}^\ell$ with respect to frame $\mathcal{F}^k$ in terms of the relative motion tensors of these two frames with respect to frame $\mathcal{F}^\mathcal{I}$.

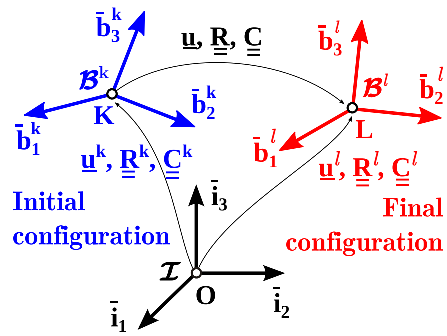

Figure 1.5: Configuration of three frames.

Introducing eqs. (1.14) and (1.20) into eq. (1.42), the components of the relative motion tensor, resolved in frame $\mathcal{F}^\mathcal{I}$, become

$$\underline{\underline{\mathcal{C}}} = \begin{bmatrix} \underline{\underline{R}}^\ell & \widetilde{u}^\ell \underline{\underline{R}}^\ell \\ \underline{\underline{0}} & \underline{\underline{R}}^\ell \end{bmatrix} \begin{bmatrix} \underline{\underline{R}}^{kT} & \underline{\underline{R}}^{kT} \widetilde{u}^{kT} \\ \underline{\underline{0}} & \underline{\underline{R}}^{kT} \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}^\ell \underline{\underline{R}}^{kT} & \widetilde{u}_O \underline{\underline{R}}^\ell \underline{\underline{R}}^{kT} \\ \underline{\underline{0}} & \underline{\underline{R}}^\ell \underline{\underline{R}}^{kT} \end{bmatrix}, \tag{1.43}$$

where $\underline{u}_O = (\underline{u}^\ell - \underline{u}^k) - (\underline{\underline{R}}^\ell \underline{\underline{R}}^{kT} - \underline{\underline{I}})\underline{u}^k$. Figure 1.5 shows that $\underline{u} = \underline{u}^\ell - \underline{u}^k$ is the relative position vector of point **L** with respect to point **K** and $\underline{\underline{R}} = \underline{\underline{R}}^\ell \underline{\underline{R}}^{kT}$ is the relative rotation of basis $\mathcal{B}^\ell$ with respect to basis $\mathcal{B}^k$. With these notations, the relative motion tensor, eq. (1.43), becomes

$$\underline{\underline{\mathcal{C}}} = \begin{bmatrix} \underline{\underline{R}} & \widetilde{u}_O \underline{\underline{R}} \\ \underline{\underline{0}} & \underline{\underline{R}} \end{bmatrix}, \tag{1.44}$$

where vector $\underline{u}_O$ is defined as

$$\underline{u}_O = \underline{u} - \left(\underline{\underline{R}} - \underline{\underline{I}}\right) \underline{u}^k. \tag{1.45}$$

The discussion thus far has focused on the abstract concept of frames. In contrast, section 1.1 deals with the general motion of rigid bodies. It is clear that a one-to-one relationship exist between a frame and the configuration a rigid body; in fact, fig. 1.5 can be interpreted as representing the configuration of a rigid body in its initial and final configurations. Consider now the material point of the body whose location coincides with that of point **O** in the initial configuration. The displacement of this material point as the body moves to its final configuration is given by eq. (1.2) as $\underline{u}_O = \underline{u} + \left(\underline{\underline{R}} - \underline{\underline{I}}\right)(-\underline{u}^k)$. This observation allows the geometric interpretation of vector $\underline{u}_O$ defined by eq. (1.45): it represents the displacement of the point of the rigid body whose location coincides with point $\mathbf{O}$ in the reference configuration.

In section 1.1.1, the motion of a rigid body was defined in an intrinsic manner by eq. (1.10). The developments presented here also give an intrinsic representation, but based on a different set of quantities. Intuitively, the two representations should be closely linked.

Starting from the representation given by eq. (1.10), the relative rotation tensor, $\underline{\underline{R}}$, is readily obtained from unit vector $\bar{n}$ and the magnitude of the rotation, ϕ. To complete the determination of the motion tensor, vector $\underline{u}_O$ must be evaluated. Equation (1.11) yields

$$\underline{u}_O = d\bar{n} + 2\sin\phi/2\ \underline{\underline{G}}\,\underline{w}. \tag{1.46}$$

Clearly, given the intrinsic definition of the motion in eq. (1.10), the motion tensor is obtained easily.

Conversely, if the motion tensor is know, unit vector $\bar{n}$ and the magnitude, ϕ, of the rotation are obtained easily. Next, vector $\underline{u}_O$ is extracted from the motion tensor and yields the intrinsic displacement of the rigid body as $d = \bar{n}^T \underline{u}_O$. Finally, the Plücker coordinates or the Mozzi-Chasles axis are found by inverting eq. (1.46) yields

$$\underline{w} = \widetilde{r}_Q \bar{n} = \underline{\underline{G}}^T \frac{\underline{u}_O - d\bar{n}}{2\sin\phi/2}. \tag{1.47}$$

By imposing the normality condition $\bar{n}^T \underline{r}_Q = 0$, the point of the Mozzi-Chasles axis that is at the shortest distance from point $\mathbf{O}$ is found as $\underline{r}_Q = \widetilde{n}\underline{\underline{G}}^T \underline{u}_O/(2\sin\phi/2)$.

The relative motion tensor defined by eq. (1.44) only involves intrinsic quantities, *i.e.*, quantities that are independent of the selection of a particular reference point of the rigid body. The components of the same relative motion tensor resolved in frame $\mathcal{F}^k$ are $\underline{\underline{\mathcal{C}}}^* = \underline{\underline{\mathcal{C}}}^{k-1}\underline{\underline{\mathcal{C}}}\,\underline{\underline{\mathcal{C}}}^k$, where notation $(\cdot)^*$ indicates tensor quantities resolved in frame $\mathcal{F}^k$. Tedious algebra reveals the following result

$$\underline{\underline{\mathcal{C}}}^* = \begin{bmatrix} \underline{\underline{R}}^{kT}\underline{\underline{R}}^{\ell} & \widetilde{u}^*\underline{\underline{R}}^{kT}\underline{\underline{R}}^{\ell} \\ \underline{\underline{0}} & \underline{\underline{R}}^{kT}\underline{\underline{R}}^{\ell} \end{bmatrix}, \tag{1.48}$$

where $\underline{u}^* = \underline{\underline{R}}^{kT}(\underline{u}^{\ell} - \underline{u}^k)$. Clearly, $\underline{\underline{R}}^{kT}\underline{\underline{R}}^{\ell} = \underline{\underline{R}}^{kT}\underline{\underline{R}}\,\underline{\underline{R}}^{k} = \underline{\underline{R}}^*$ are the components of the relative rotation tensor resolved in basis $\mathcal{B}^k$. With these notations, the components of the relative motion tensor resolved in the material frame become

$$\underline{\underline{\mathcal{C}}}^* = \begin{bmatrix} \underline{\underline{R}}^* & \widetilde{u}^*\underline{\underline{R}}^* \\ \underline{\underline{0}} & \underline{\underline{R}}^* \end{bmatrix}. \tag{1.49}$$

This expession involves two quantities. First, the components of the relative rotation tensor resolved in basis

$\mathcal{B}^k$, as expected, and second, the components of the displacement vector of point **A**, also resolved in basis $\mathcal{B}^k$. Clearly, this is not an intrinsic expression because it involves the components of the displacement vector of a *specific point of the rigid body.*

1.3 Velocity field of a rigid body

The time-dependent motion of a rigid body, as depicted in fig. 1.6, will now be investigated. The structure of the velocity field of the entire rigid body is the focus of the analysis.

The inertial velocity of material point **P** is obtained from a time derivative of eq. (1.2), $\underline{v}_P = \underline{v} + \dot{\underline{\underline{R}}}\,\underline{s}_P = \underline{v} + \dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T \underline{S}_P$, where $\underline{v}_P = \dot{\underline{u}}_P$ and $\underline{v} = \dot{\underline{u}}$ are the inertial velocity vectors of point **P** and **A**, respectively. This equation becomes

$$\underline{v}_P = \underline{v} + \widetilde{\omega}\underline{S}_P, \tag{1.50}$$

where $\underline{\omega} = \text{axial}(\dot{\underline{\underline{R}}}\,\underline{\underline{R}}^T)$ is the angular velocity vector of the rigid body. This relationship describes the velocity of an arbitrary point **P** of the rigid body in terms of $\underline{v}$, the velocity of a reference point, and $\underline{\omega}$, the angular velocity vector of the rigid body. Here again, the choice of reference point **A** is arbitrary, and hence, eq. (1.50) is not an intrinsic relationship.

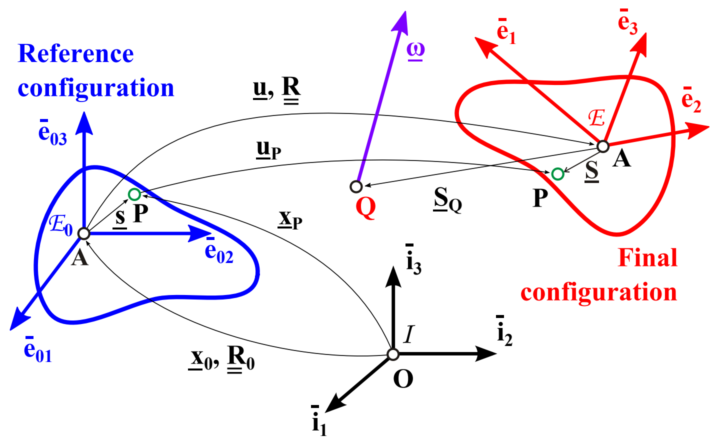

Figure 1.6: Time-dependent motion of a rigid body.

To obtain a more general description of the velocity field, the following question can be asked: is it possible to find a material point of the rigid body, say point **Q**, whose velocity vector is parallel to the angular velocity vector? If such a point exists, the following relationship must hold

$$\underline{v}_Q = \underline{v} + \widetilde{\omega}\underline{S}_Q = \mu\underline{\omega}, \tag{1.51}$$

where μ is an arbitrary scalar that can be found by taking the scalar product of this equation by $\underline{\omega}^T$ to find $\mu = (\underline{\omega}^T \underline{v})/\omega^2$.

Equation (1.51) now becomes $\widetilde{\omega}\underline{S}_Q = (\underline{\omega}\,\underline{\omega}^T/\omega^2 - \underline{\underline{I}})\underline{v} = \widetilde{\omega}\widetilde{\omega}\underline{v}/\omega^2$. This equation can be recast as $\widetilde{\omega}\left[\underline{S}_Q - \widetilde{\omega}\underline{v}/\omega^2\right] = 0$. The bracketed term is parallel to the angular velocity vector, which implies $\underline{S}_Q - \widetilde{\omega}\underline{v}/\omega^2 =$

$\alpha\underline{\omega}$, where α is an arbitrary constant. The location of point $\mathbf{Q}$ is now found as

$$\underline{S}_Q = \alpha\underline{\omega} + \frac{\widetilde{\omega}}{\omega^2}\underline{v}.$$

The solution is the locus of points along a straight line parallel to $\underline{\omega}$, and hence, no unique solution exists for the location of point $\mathbf{Q}$.

To remove this ambiguity, point $\mathbf{Q}$ will be selected as that at the shortest distance from point $\mathbf{A}$, *i.e.*, $\underline{\omega}^T\underline{S}_Q = 0$. It follows that $\alpha = 0$, and

$$\underline{S}_Q = \frac{\widetilde{\omega}}{\omega^2}\underline{v}. \tag{1.52}$$

In summary, material point $\mathbf{Q}$ of the rigid body exists whose velocity vector is parallel to the angular velocity vector. The location of this point is given by eq. (1.52). Combining eqs. (1.50) and (1.51) now yields

$$\underline{v}_P = \frac{\underline{\omega}^T\underline{v}}{\omega^2}\underline{\omega} + \widetilde{\omega}(\underline{S}_P - \underline{S}_Q) = \underline{v}_Q + \widetilde{\omega}(\underline{S}_P - \underline{S}_Q) \tag{1.53}$$

This relationship expresses the velocity of material point $\mathbf{P}$ of the rigid body as the velocity of point $\mathbf{Q}$, $\underline{v}_Q$, which is parallel to angular velocity vector $\underline{\omega}$, followed by a rotation about that same axis. This is referred to as *screw motion* about axis $\underline{\omega}$. The *screw axis* is defined as the line passing through point $\mathbf{Q}$ and parallel to $\underline{\omega}$. The Plücker coordinates, $\underline{\mathcal{Q}}$, of the screw axis are

$$\underline{\mathcal{Q}}_{SA} = \begin{Bmatrix} -\dfrac{\widetilde{\omega}\widetilde{\omega}}{\omega^2}\underline{v} \\ \underline{\omega} \end{Bmatrix} \tag{1.54}$$

1.4 Derivatives of finite motion operations

The derivatives of finite rotation operations lead to the concept of angular velocity vector. The present section focuses on the study of time derivatives of the motion tensor, which leads to both velocity and angular velocity vectors. Differential changes in motion are also investigated.

1.4.1 The velocity vector

The time-dependent motion of a rigid body is represented by the time-dependent motion of the body attached frame, $\mathcal{F} = \left[\mathbf{A}, \mathcal{B}^* = (\bar{b}_1, \bar{b}_3, \bar{b}_3)\right]$, depicted in fig. 1.3. Let $\underline{\underline{\mathcal{C}}}$ be the motion tensor that brings reference frame $\mathcal{F}^I$ to frame $\mathcal{F}$, and eq. (1.15) then implies $\underline{\mathcal{Q}}(t) = \underline{\underline{\mathcal{C}}}(t)\underline{\mathcal{Q}}^*$. Taking a time derivative of this equation leads to

$\dot{\underline{\mathcal{Q}}} = \dot{\underline{\underline{\mathcal{C}}}}\, \underline{\mathcal{Q}}^*$, and eliminating $\underline{\mathcal{Q}}^*$ then yields

$$\dot{\underline{\mathcal{Q}}} = \dot{\underline{\underline{\mathcal{C}}}}\, \underline{\underline{\mathcal{C}}}^{-1} \underline{\mathcal{Q}}. \tag{1.55}$$

The use of well-known identity leads to

$$\dot{\underline{\underline{\mathcal{C}}}}\, \underline{\underline{\mathcal{C}}}^{-1} = \begin{bmatrix} \dot{\underline{\underline{R}}} & \dot{\widetilde{u}}\underline{\underline{R}} + \widetilde{u}\dot{\underline{\underline{R}}} \\ \underline{\underline{0}} & \dot{\underline{\underline{R}}} \end{bmatrix} \begin{bmatrix} \underline{\underline{R}}^T & \underline{\underline{R}}^T \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{R}}^T \end{bmatrix} = \begin{bmatrix} \widetilde{\omega} & (\widetilde{\dot{\underline{u}} + \widetilde{u}\underline{\omega}}) \\ \underline{\underline{0}} & \widetilde{\omega} \end{bmatrix} = \begin{bmatrix} \widetilde{\omega} & \widetilde{v} \\ \underline{\underline{0}} & \widetilde{\omega} \end{bmatrix}. \tag{1.56}$$

This expression gives rise to two quantities. First, the angular velocity of the rigid body emerges from the time derivative of the rotation tensor, $\underline{\omega} = \text{axial}(\dot{\underline{\underline{R}}}\, \underline{\underline{R}}^T)$; as expected, this quantity is identical to that which arose for the study of time derivatives of time-dependent rotations. Second, the velocity vector of the rigid body, $\underline{v} = \dot{\underline{u}} + \widetilde{u}\underline{\omega}$, also emerges from the time derivative of the motion tensor. This quantity can be interpreted as the linear velocity of the point of the rigid body that instantaneously coincides with the origin of the reference frame, point **O**.

The velocity vector of the rigid body resolved in frame $\mathcal{F}^I$ is now defined as

$$\underline{\mathcal{V}} = \begin{Bmatrix} \underline{v} \\ \underline{\omega} \end{Bmatrix}, \tag{1.57}$$

and eq. (1.55) becomes $\dot{\underline{\mathcal{Q}}} = \widetilde{\mathcal{V}}\underline{\mathcal{Q}}$, where the generalized vector product tensor is given by eq. (1.35).

It is also possible to resolve the components of the velocity vector in the moving frame,

$$\underline{\underline{\mathcal{C}}}^{-1} \dot{\underline{\mathcal{Q}}} = \underline{\underline{\mathcal{C}}}^{-1} \dot{\underline{\underline{\mathcal{C}}}}\, \underline{\mathcal{Q}}^*. \tag{1.58}$$

It is readily found that

$$\underline{\underline{\mathcal{C}}}^{-1} \dot{\underline{\underline{\mathcal{C}}}} = \begin{bmatrix} \underline{\underline{R}}^T & \underline{\underline{R}}^T \widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{R}}^T \end{bmatrix} \begin{bmatrix} \dot{\underline{\underline{R}}} & \dot{\widetilde{u}}\underline{\underline{R}} + \widetilde{u}\dot{\underline{\underline{R}}} \\ \underline{\underline{0}} & \dot{\underline{\underline{R}}} \end{bmatrix} = \begin{bmatrix} \widetilde{\omega}^* & \widetilde{\underline{\underline{R}}^T \dot{\underline{u}}} \\ \underline{\underline{0}} & \widetilde{\omega} \end{bmatrix} = \begin{bmatrix} \widetilde{\omega}^* & \widetilde{v}^* \\ \underline{\underline{0}} & \widetilde{\omega}^* \end{bmatrix}. \tag{1.59}$$

This expression gives rise to two quantities. First, the components of the angular velocity of the rigid body resolved in the rotating basis, $\underline{\omega}^* = \text{axial}(\underline{\underline{R}}^T \dot{\underline{\underline{R}}})$. Second, the components of the velocity vector of the reference point of rigid body resolved in the rotating basis, $\underline{v}^* = \underline{\underline{R}}^T \dot{\underline{u}}$.

The components of the velocity vector of the rigid body resolved in the material frame are now defined as

$$\underline{\mathcal{V}}^* = \begin{Bmatrix} \underline{v}^* \\ \underline{\omega}^* \end{Bmatrix} = \underline{\underline{\mathcal{C}}}^{-1}\underline{\mathcal{V}}. \tag{1.60}$$

Equation (1.58) now becomes $\underline{\underline{\mathcal{C}}}^{-1}\dot{\underline{\mathcal{Q}}} = \widetilde{\mathcal{V}}^*\underline{\mathcal{Q}}^*$, where the generalized vector product operator is given by eq. (1.35).

The above developments are summarized in the following relationships

$$\dot{\underline{\underline{\mathcal{C}}}}\,\underline{\underline{\mathcal{C}}}^{-1} = \widetilde{\mathcal{V}}, \qquad \underline{\underline{\mathcal{C}}}\,\dot{\underline{\underline{\mathcal{C}}}}^{-1} = -\widetilde{\mathcal{V}}, \tag{1.61a}$$

$$\underline{\underline{\mathcal{C}}}^{-1}\dot{\underline{\underline{\mathcal{C}}}} = \widetilde{\mathcal{V}}^*, \quad \dot{\underline{\underline{\mathcal{C}}}}^{-1}\underline{\underline{\mathcal{C}}} = -\widetilde{\mathcal{V}}^*. \tag{1.61b}$$

As expected, it is readily shown that

$$\widetilde{\mathcal{V}}^* = \underline{\underline{\mathcal{C}}}^{-1}\widetilde{\mathcal{V}}\underline{\underline{\mathcal{C}}}, \tag{1.62a}$$

$$\widetilde{\mathcal{V}} = \underline{\underline{\mathcal{C}}}\,\widetilde{\mathcal{V}}^*\underline{\underline{\mathcal{C}}}^{-1}. \tag{1.62b}$$

1.4.2 The differential motion vector

The concept of differential rotation vector was introduced based on the rotation tensor. By analogy, the following expression is formed

$$\mathrm{d}\underline{\underline{\mathcal{C}}}\,\underline{\underline{\mathcal{C}}}^{-1} = \begin{bmatrix} \mathrm{d}\underline{\underline{R}} & \widetilde{\mathrm{d}u}\,\underline{\underline{R}} + \widetilde{u}\,\mathrm{d}\underline{\underline{R}} \\ \underline{\underline{0}} & \mathrm{d}\underline{\underline{R}} \end{bmatrix} \begin{bmatrix} \underline{\underline{R}}^T & \underline{\underline{R}}^T\widetilde{u}^T \\ \underline{\underline{0}} & \underline{\underline{R}}^T \end{bmatrix} = \begin{bmatrix} \widetilde{\mathrm{d}\psi} & (\widetilde{\mathrm{d}\underline{u} + \widetilde{u}\underline{\mathrm{d}\psi}}) \\ \underline{\underline{0}} & \widetilde{\mathrm{d}\psi} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathrm{d}\psi} & \widetilde{\mathrm{d}u} \\ \underline{\underline{0}} & \widetilde{\mathrm{d}\psi} \end{bmatrix}.$$

This expression gives rise to two quantities. First, the differential rotation vector of the rigid body emerges from differential changes of the rotation tensor, $\underline{\mathrm{d}\psi} = \mathrm{axial}(\mathrm{d}\underline{\underline{R}}\,\underline{\underline{R}}^T)$. No vector $\underline{\psi}$ exists such that $\mathrm{d}(\underline{\psi})$ gives the differential rotation vector.

Second, the differential displacement vector of the rigid body, $\underline{\mathrm{d}u} = \mathrm{d}\underline{u} + \widetilde{u}\,\underline{\mathrm{d}\psi}$, also emerges from the differential of the motion tensor. $\mathrm{d}\underline{u}$ is the differential displacement of point **A** and $\underline{\mathrm{d}u} = \mathrm{d}\underline{u} + \widetilde{u}\,\underline{\mathrm{d}\psi}$ the differential displacement of the material point of the rigid body that instantaneously coincides with point **O**. Of course, there exist no displacement vector, say $\underline{x}$, such $\mathrm{d}(\underline{x}) = \mathrm{d}\underline{u} + \widetilde{u}\,\underline{\mathrm{d}\psi}$. Notations $\underline{\mathrm{d}u}$ and $\underline{\mathrm{d}\psi}$ will be used to denote the differential displacement and rotation vectors, respectively.

By analogy to eqs. (1.61a) and (1.61b), the following compact notation is adopted

$$\mathrm{d}\underline{\underline{\mathcal{C}}}\,\underline{\underline{\mathcal{C}}}^{-1} = \mathrm{d}\widetilde{\mathcal{U}}, \qquad \underline{\underline{\mathcal{C}}}\mathrm{d}\underline{\underline{\mathcal{C}}}^{-1} = -\mathrm{d}\widetilde{\mathcal{U}}, \tag{1.63a}$$

$$\underline{\underline{\mathcal{C}}}^{-1}\mathrm{d}\underline{\underline{\mathcal{C}}} = \mathrm{d}\widetilde{\mathcal{U}}^{*}, \quad \mathrm{d}\underline{\underline{\mathcal{C}}}^{-1}\underline{\underline{\mathcal{C}}} = -\mathrm{d}\widetilde{\mathcal{U}}^{*}, \tag{1.63b}$$

where the components of the *differential motion vector* are defined as

$$\underline{\mathrm{d}\mathcal{U}} = \begin{Bmatrix} \underline{\mathrm{d}u} \\ \underline{\mathrm{d}\psi} \end{Bmatrix} = \underline{\underline{\mathcal{C}}}\,\underline{\mathrm{d}\mathcal{U}}^{*}, \tag{1.64a}$$

$$\underline{\mathrm{d}\mathcal{U}}^{*} = \begin{Bmatrix} \underline{\mathrm{d}u}^{*} \\ \underline{\mathrm{d}\psi}^{*} \end{Bmatrix} = \underline{\underline{\mathcal{C}}}^{-1}\underline{\mathrm{d}\mathcal{U}}, \tag{1.64b}$$

in the fixed and moving frames, respectively. The components of the differential rotation and displacement vectors, both resolved in the moving frame, are $\underline{\mathrm{d}\psi}^{*} = \mathrm{axial}(\underline{\underline{R}}^{T}\mathrm{d}\underline{\underline{R}})$ and $\underline{\mathrm{d}u}^{*} = \underline{\underline{R}}^{T}\mathrm{d}\underline{u}$, respectively.

It is readily shown that

$$\mathrm{d}\widetilde{\mathcal{U}}^{*} = \underline{\underline{\mathcal{C}}}^{-1}\mathrm{d}\widetilde{\mathcal{U}}\underline{\underline{\mathcal{C}}}, \tag{1.65a}$$

$$\mathrm{d}\widetilde{\mathcal{U}} = \underline{\underline{\mathcal{C}}}\,\mathrm{d}\widetilde{\mathcal{U}}^{*}\underline{\underline{\mathcal{C}}}^{-1}. \tag{1.65b}$$

Taking a differential of eq. (1.61a) and a time derivative of eq. (1.63a) leads to $\mathrm{d}\widetilde{\mathcal{V}} = \mathrm{d}\dot{\underline{\underline{\mathcal{C}}}}\,\underline{\underline{\mathcal{C}}}^{-1} + \dot{\underline{\underline{\mathcal{C}}}}\mathrm{d}\underline{\underline{\mathcal{C}}}^{-1}$ and $\dot{\widetilde{\mathrm{d}\mathcal{U}}} = \mathrm{d}\dot{\underline{\underline{\mathcal{C}}}}\,\underline{\underline{\mathcal{C}}}^{-1} + \mathrm{d}\underline{\underline{\mathcal{C}}}\,\dot{\underline{\underline{\mathcal{C}}}}^{-1}$, respectively. Subtracting these two equations and using eqs. (1.61a) and (1.63a) then yields

$$\mathrm{d}\widetilde{\mathcal{V}} - \dot{\widetilde{\mathrm{d}\mathcal{U}}} = -\widetilde{\mathcal{V}}\,\mathrm{d}\widetilde{\mathcal{U}} + \mathrm{d}\widetilde{\mathcal{U}}\,\widetilde{\mathcal{V}}.$$

Expanding these expressions and using well-known identities then leads to this important result $\mathrm{d}\underline{\mathcal{V}} = \underline{\mathrm{d}\dot{\mathcal{U}}} - \widetilde{\mathcal{V}}\underline{\mathrm{d}\mathcal{U}}$, which relates differentials in the velocity vector to the differential motion vector and its time derivative.

The following results are obtained in a similar manner

$$\mathrm{d}\underline{\mathcal{V}} = \underline{\mathrm{d}\dot{\mathcal{U}}} - \widetilde{\mathcal{V}}\underline{\mathrm{d}\mathcal{U}}, \qquad \mathrm{d}\underline{\mathcal{V}} = \underline{\underline{\mathcal{C}}}\,\underline{\mathrm{d}\dot{\mathcal{U}}}^{*}, \tag{1.66a}$$

$$\mathrm{d}\underline{\mathcal{V}}^{*} = \underline{\mathrm{d}\dot{\mathcal{U}}}^{*} + \widetilde{\mathcal{V}}^{*}\underline{\mathrm{d}\mathcal{U}}^{*}, \quad \mathrm{d}\underline{\mathcal{V}}^{*} = \underline{\underline{\mathcal{C}}}^{-1}\underline{\mathrm{d}\dot{\mathcal{U}}}. \tag{1.66b}$$

1.5 Time derivatives and variation of rigid body motion operations

Consider a fixed frame $\mathcal{F}^{\mathcal{I}} = [\mathbf{O}, \mathcal{I} = (\bar{\imath}_i, \bar{\imath}_2, \bar{\imath}_3)]$, and a time-dependent frame $\mathcal{F}^k = [\mathbf{K}, \mathcal{B}^k = (\bar{b}_1^k, \bar{b}_2^k \bar{b}_3^k)]$. It's often the case that the motion of this time-dependent frame depend on a scalar variable, say time t. If $\underline{\underline{\mathcal{C}}}(t)$ is the time-dependent motion tensor that bring frame $\mathcal{F}^{\mathcal{I}}$ to $\mathcal{F}^k$, ${}^k\bar{\mathcal{Y}}_i(t) = \underline{\underline{\mathcal{C}}}^k(t)^{\mathcal{I}}\bar{\mathcal{Y}}_i$, $i = 1, 2, \cdots, 6$. The time derivative and variation of this expression are ${}^k\dot{\bar{\mathcal{Y}}}_i(t) = \dot{\underline{\underline{\mathcal{C}}}}(t)^{\mathcal{I}}\bar{\mathcal{Y}}_i = \dot{\underline{\underline{\mathcal{C}}}}(t)\underline{\underline{\mathcal{C}}}(t)^{-1\,k}\bar{\mathcal{Y}}_i$, and $\delta\,{}^k\bar{\mathcal{Y}}_i(t) = \delta\underline{\underline{\mathcal{C}}}(t)^{\mathcal{I}}\bar{\mathcal{Y}}_i = \delta\underline{\underline{\mathcal{C}}}(t)\underline{\underline{\mathcal{C}}}(t)^{-1\,k}\bar{\mathcal{Y}}_i$, where notation $\dot{(\cdot)}$ and $\delta(\cdot)$ indicates a time derivative and variation, respectively.

The *generalized velocity vector* of a rigid body motion is now defined as

$$\underline{\mathcal{V}} \times \underline{\underline{\mathcal{I}}} = \dot{\underline{\underline{\mathcal{C}}}}(t)\underline{\underline{\mathcal{C}}}(t)^{-1} = ({}^k\dot{\bar{\mathcal{Y}}}_i(t) \otimes {}^{\mathcal{I}}\bar{\mathcal{Y}}^i)({}^k\bar{\mathcal{Y}}_i(t) \otimes {}^{\mathcal{I}}\bar{\mathcal{Y}}^i)^{-1} = {}^k\dot{\bar{\mathcal{Y}}}_i(t) \otimes {}^k\bar{\mathcal{Y}}^i(t) \tag{1.67}$$

where $\underline{\mathcal{V}} = \begin{Bmatrix} \underline{\mathcal{V}}_1 \\ \underline{\mathcal{V}}_2 \end{Bmatrix}$, $\underline{\underline{\mathcal{I}}} = {}^{[\mathcal{I}]}\bar{\mathcal{Y}}_i \otimes {}^{[\mathcal{I}]}\underline{\mathcal{Y}}^i = {}^k\bar{\mathcal{Y}}_i(t) \otimes {}^k\bar{\mathcal{Y}}^i(t)$ is the identity tensor, and $\underline{\mathcal{V}} \times \underline{\underline{\mathcal{I}}} = \begin{bmatrix} \underline{\mathcal{V}}_2 \times \underline{\underline{I}} & \underline{\mathcal{V}}_1 \times \underline{\underline{I}} \\ \underline{\underline{0}} & \underline{\mathcal{V}}_2 \times \underline{\underline{I}} \end{bmatrix}$.

1.6 Relationships between motion tensor and screw

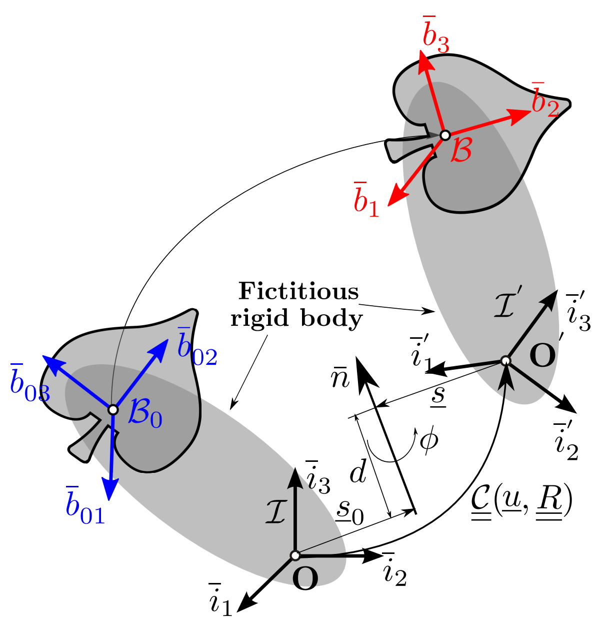

Figure 1.7: Screw.

Consider a rigid motion as depicted in Fig.(1.7). The screw corresponding to this rigid motion can be expressed as follows

$$\underline{\mathcal{S}} = \begin{Bmatrix} \underline{q} \\ \underline{p} \end{Bmatrix} = p \begin{Bmatrix} \hat{\underline{q}} \\ \bar{n} \end{Bmatrix} = p \begin{Bmatrix} \widetilde{s}_0 \bar{n} + \alpha\, d\bar{n} \\ \bar{n} \end{Bmatrix} \tag{1.68}$$

where, $p = f(\phi)$, $d = \hat{\underline{q}}^T \bar{n}/\alpha$, $\underline{s}_0 = \dfrac{\widetilde{n}\underline{\underline{G}}^T \underline{u}}{2\sin\phi/2}$, ϕ is the rotational angel, and α is an arbitrary coefficient except for $\alpha = 0$. It's clear that a screw $\underline{\mathcal{S}}$ has 6 degrees of freedom, therefore a rigid motion can be represented by a screw (a line vector or the Plücker coordinate of a line only has 4 degrees of freedom). Vector $\bar{n}$ is the axis of a finite rotation, and p is a odd function of angel ϕ.

p and d are invariant under frame transformation. Let $\underline{\mathcal{S}}^*$ denote the components of a screw resolved in frame $\mathcal{F}^k$, and it's components resolved in frame $\mathcal{F}^{\mathcal{I}}$ can be obtain as following

$$\underline{\mathcal{S}}^{[\mathcal{I}]} = {}^k\underline{\underline{\mathcal{C}}}^{[\mathcal{I}]}\underline{\mathcal{S}}^* = \begin{bmatrix} \underline{\underline{R}}^k & \widetilde{u}^k \underline{\underline{R}}^k \\ \underline{\underline{0}} & \underline{\underline{R}}^k \end{bmatrix} \begin{Bmatrix} \underline{q}^* \\ \underline{p}^* \end{Bmatrix} = \begin{Bmatrix} \widetilde{u}^k \underline{\underline{R}}^k\, \underline{p}^* + \underline{\underline{R}}^k\, \underline{q}^* \\ \underline{\underline{R}}^k \underline{p}^* \end{Bmatrix} = \begin{Bmatrix} \underline{q}^{[\mathcal{I}]} \\ \underline{p}^{[\mathcal{I}]} \end{Bmatrix} \tag{1.69}$$

From eq.(1.69), we can obtain

$$p^{'} = \|\underline{p}^{[\mathcal{I}]}\| = \|\underline{\underline{R}}^k \underline{p}^*\| = \|\underline{p}^*\| = p \tag{1.70}$$

$$\alpha\, d^{'} = \underline{p}^{[\mathcal{I}]^T} \underline{q}^{[\mathcal{I}]} = (\underline{\underline{R}}^k\, \underline{p}^*)^T (\widetilde{u}^k \underline{\underline{R}}^k\, \underline{p}^* + \underline{\underline{R}}^k\, \underline{q}^*)/p^{'} = \alpha\, d \tag{1.71}$$

Two linearly independent eigenvectors of $\underline{\underline{\mathcal{C}}}^{[\mathcal{I}]}$ associated with its unit eigenvalues are found to be

$$\underline{\mathcal{N}}_1^{\dagger} = \begin{Bmatrix} \bar{n} \\ \underline{0} \end{Bmatrix}, \quad \textit{and} \quad \underline{\mathcal{N}}_2^{\dagger} = \begin{Bmatrix} \dfrac{-\widetilde{n}\widetilde{n}\underline{\underline{G}}^T \underline{r}}{2\sin\phi/2} \\ \underline{0} \end{Bmatrix} \tag{1.72}$$

The family of eigenvectors associated with the unit eigenvalues can be expressed as follows

$$\underline{\mathcal{N}} = \frac{\alpha}{2\sin\phi/2}\underline{\mathcal{N}}_1^{\dagger} + \underline{\mathcal{N}}_2^{\dagger} \tag{1.73}$$

It's clear that $p\underline{\mathcal{N}}$ is a screw.

1.7 Transitivity equations of rigid body motion

If we assume $\delta\underline{\underline{\dot{\mathcal{C}}}} = \dot{\delta\underline{\underline{\mathcal{C}}}}$, or equivalently $\delta\underline{\dot{q}} = \dot{\delta\underline{q}}$ ($\underline{q}$ are the motion parameters), then the following results are obtained

$$\delta\underline{\mathcal{V}} = \underline{\dot{\delta\mathcal{U}}} - \widetilde{\mathcal{V}}\underline{\delta\mathcal{U}}, \qquad \delta\underline{\mathcal{V}} = \underline{\underline{\mathcal{C}}}\,\underline{\dot{\delta\mathcal{U}}}^{*} \tag{1.74}$$

$$\delta\underline{\mathcal{V}}^{*} = \underline{\dot{\delta\mathcal{U}}}^{*} + \widetilde{\mathcal{V}}^{*}\underline{\delta\mathcal{U}}^{*}, \qquad \delta\underline{\mathcal{V}}^{*} = \underline{\underline{\mathcal{C}}}^{-1}\underline{\dot{\delta\mathcal{U}}} \tag{1.75}$$

Variations of $\underline{\mathcal{V}} = \underline{\mathcal{V}}(\underline{\dot{q}}, \underline{q}, t)$, $\underline{\mathcal{V}}^{*} = \underline{\mathcal{V}}^{*}(\underline{\dot{q}}, \underline{q}, t)$ can be expressed as following

$$\delta\underline{\mathcal{V}} = \frac{\partial\underline{\mathcal{V}}}{\partial\underline{\dot{q}}}\delta\underline{\dot{q}} + \frac{\partial\underline{\mathcal{V}}}{\partial\underline{q}}\delta\underline{q} \tag{1.76}$$

$$\delta\underline{\mathcal{V}}^{*} = \frac{\partial\underline{\mathcal{V}}^{*}}{\partial\underline{\dot{q}}}\delta\underline{\dot{q}} + \frac{\partial\underline{\mathcal{V}}^{*}}{\partial\underline{q}}\delta\underline{q} \tag{1.77}$$

The time derivative of $\underline{\delta\mathcal{U}} = \dfrac{\partial\underline{\mathcal{V}}}{\partial\underline{\dot{q}}}\delta\underline{q}$, $\underline{\delta\mathcal{U}}^{*} = \dfrac{\partial\underline{\mathcal{V}}^{*}}{\partial\underline{\dot{q}}}\delta\underline{q}$ can be expressed as following

$$\underline{\dot{\delta\mathcal{U}}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\underline{\mathcal{V}}}{\partial\underline{\dot{q}}}\delta\underline{q} + \frac{\partial\underline{\mathcal{V}}}{\partial\underline{\dot{q}}}\dot{\delta\underline{q}} \tag{1.78}$$

$$\underline{\dot{\delta\mathcal{U}}}^{*} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\underline{\mathcal{V}}^{*}}{\partial\underline{\dot{q}}}\delta\underline{q} + \frac{\partial\underline{\mathcal{V}}^{*}}{\partial\underline{\dot{q}}}\dot{\delta\underline{q}} \tag{1.79}$$

subtracting eqs. (1.76) with (1.78), (1.77) with (1.79), and note that $\delta\underline{\dot{q}} = \dot{\delta\underline{q}}$, we obtain the following transitivity equations

$$\underline{\dot{\delta\mathcal{U}}} - \delta\underline{\mathcal{V}} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\underline{\mathcal{V}}}{\partial\underline{\dot{q}}} - \frac{\partial\underline{\mathcal{V}}}{\partial\underline{q}}\right)\delta\underline{q} \tag{1.80}$$

$$\underline{\dot{\delta\mathcal{U}}}^{*} - \delta\underline{\mathcal{V}}^{*} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\underline{\mathcal{V}}^{*}}{\partial\underline{\dot{q}}} - \frac{\partial\underline{\mathcal{V}}^{*}}{\partial\underline{q}}\right)\delta\underline{q} \tag{1.81}$$

Substituting eqs. (1.76), (1.77), (1.78), and (1.79) into eqs. (1.74) and (1.75), and note that $\underline{\underline{\mathcal{H}}} = \partial\underline{\mathcal{V}}/\partial\underline{\dot{q}}$, and $\underline{\underline{\mathcal{H}}}^{*} = \partial\underline{\mathcal{V}}^{*}/\partial\underline{\dot{q}}$, we obtain the following two sets of transitivity equations

$$\underline{\underline{\dot{\mathcal{H}}}} - \frac{\partial\underline{\mathcal{V}}}{\partial\underline{q}} = \widetilde{\mathcal{V}}\underline{\underline{\mathcal{H}}} \tag{1.82}$$

$$\underline{\underline{\dot{\mathcal{H}}}}^{*} - \frac{\partial\underline{\mathcal{V}}^{*}}{\partial\underline{q}} = -\widetilde{\mathcal{V}}^{*}\underline{\underline{\mathcal{H}}} \tag{1.83}$$

$$\underline{\underline{\mathcal{C}}}^{-1}\dot{\underline{\underline{\mathcal{H}}}} - \frac{\partial \underline{\mathcal{V}}^*}{\partial \underline{q}} = \underline{0} \tag{1.84}$$

$$\underline{\underline{\mathcal{C}}}\,\dot{\underline{\underline{H}}}^* - \frac{\partial \underline{\mathcal{V}}}{\partial \underline{q}} = \underline{0} \tag{1.85}$$

Chapter 2

Various facts

2.1 Notational conventions

Several notational conventions are used in the literature to denote vectors and tensors. Three widely used notations, the *geometric notation*, the *matrix notation*, and the *index notation* (?) are presented in table 2.1. The *geometric notation* is widely used in the literature, sometimes the boldface notation for vectors is replaced by a specific "vector" superscript: $\overrightarrow{a}$. The index notation is frequently used, specially when higher-order tensors must be manipulated such as in the theory of elasticity. It is, however, less often used in kinematics and dynamics.

The matrix notation is a convenient mnemonic notation and will be used exclusively in this book. Vectors are denoted with an underline, $\underline{u}$, but unit vectors are simply denoted $\bar{n}$, rather than the more cumbersome $\underline{\bar{n}}$. Tensors are denoted by a double underline, $\underline{\underline{A}}$, but skew-symmetric tensors are denoted $\widetilde{a}$, rather than the more cumbersome $\underline{\underline{\widetilde{a}}}$. Note that the tensor product, $\underline{u}\,\underline{v}^T$, also yields a tensor.

Table 2.1: The geometric, matrix, and index notations for vectors and tensors.

	Geometric notation	*Matrix notation*	*Index notation*
vector	$\mathbf{a}$	$\underline{a}$	a_i
tensor	$\underline{\underline{A}}$	$\underline{\underline{A}}$	A_{ij}
scalar product	$\mathbf{u} \cdot \mathbf{v}$	$\underline{u}^T\,\underline{v}$	$u_i v_i$
vector product	$\mathbf{u} \times \mathbf{v}$	$\widetilde{u}\underline{v}$	$u_i v_j \epsilon_{ijk}$
tensor product	$\mathbf{u} \otimes \mathbf{v}$	$\underline{u}\,\underline{v}^T$	$u_i v_j$

In practical situations, such computer implementations, it will be necessary to work with the components of specific tensors resolved in various bases. In such cases, the following notation will be used

$$\underline{a}^{[\mathcal{I}]} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix},$$

where a_1, a_2, and a_3 are the components of vector $\underline{a}$ resolved in basis $\mathcal{I}$. Because the notation $\underline{a}^{[\mathcal{I}]}$ is rather cumbersome, it will be used only when necessary; for instance, when the components of a vector in two different bases are used in the same context. When there is no possible confusion, the notation $\underline{a}^{[\mathcal{I}]}$ will be simplified as $\underline{a}$, thereby blurring the distinction between a vector and its components in a given basis.

Chapter 3

Conclusions and future work

3.1 Conclusions

A novel approach has been proposed for parallel computation in flexible multibody dynamics. The approach relies on two distinct strategies for the enforcement of the kinematic constraints at the interface between sub-domains. The traditional approach is to use global Lagrange multipliers to enforce all constraints. In the proposed approach, a hybrid strategy is used: some constraint are enforced using local Lagrange multipliers, while the remaining are imposed via global Lagrange multipliers. A coarse mesh is defined as a byproduct of the local Lagrange multiplier technique. Furthermore, an augmented Lagrangian formulation is used in conjunction with with the local Lagrange multipliers. If all kinematic constraints are enforced via this technique, the penalty terms stemming from the augmented Lagrangian formulation provide a natural conditioning of the interface problem expressed in terms of the local Lagrange multipliers. In fact, as the penalty factor increases, the condition number of the interface problem flexibility matrix tend to unity. Clearly, this approach is ideally suited for iterative solutions of the interface problem. This advantage, however, comes at the expense of the solution of a large sized coarse mesh problem. When the proposed combination of global and local Lagrange multipliers is used, it is still possible to obtain an interface problem expressed in terms of the sole global Lagrange multipliers and the solution of the coarse mesh problem provides a natural preconditioning of this interface problem.

3.2 Future work

This is an important section. Discuss possible extensions of your work and future research directions.

Acknowledgments

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