Second-Order Equations and Boundary Value Problems

Basic Quantities

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded, connected, open set.

The boundary of Ω is denoted $\partial\Omega$.

Define

$$Lu := -\operatorname{div}\big(p(x)\operatorname{grad} u\big) + q(x)u, \qquad\qquad x \in \Omega,$$

where

- $p, q \colon \Omega \to \mathbb{R}$ are sufficiently smooth
- $p(x) > 0$ for all $x \in \Omega$
- $q(x) \geq 0$ for all $x \in \Omega$

Basic Quantities

Let $I \subset \mathbb{R}$ be an open interval and

$$F\colon \Omega \to \mathbb{R} \qquad \text{or} \qquad F\colon \underset{\text{position } x}{\Omega} \times \underset{\text{time } t}{I} \to \mathbb{R}$$

be a forcing function.

Let

$$\varrho\colon \Omega \to [0, \infty) \subset \mathbb{R}$$

Depending on context,

$$u\colon \Omega \to \mathbb{C} \qquad \text{or} \qquad u\colon \Omega \times I \to \mathbb{C}$$

Second-Order Equations

- Elliptic equation

$$Lu = \varrho(x)F(x), \qquad x \in \Omega,$$

- Parabolic equation

$$\varrho(x)\frac{\partial u}{\partial t} + Lu = \varrho(x)F(x,t), \qquad (x,t) \in \Omega \times I,$$

- Hyperbolic equation

$$\varrho(x)\frac{\partial^2 u}{\partial t^2} + Lu = \varrho(x)F(x,t) \qquad (x,t) \in \Omega \times I.$$

Boundary Conditions

Boundary operator

$$Bu := \alpha(x)u + \beta(x)\frac{\partial u}{\partial n}\Big|_{\partial\Omega}$$

where $\alpha, \beta\colon \partial\Omega \to \mathbb{R}$ with

$$\alpha(x) \geq 0, \qquad \beta(x) \geq 0, \qquad \alpha(x) + \beta(x) > 0 \qquad \text{on } \partial\Omega.$$

Special cases: boundary conditions

- of the first kind (Dirichlet): $\beta(x) = 0$ for all x
- of the second kind (Neumann): $\alpha(x) = 0$ for all x
- of the third kind (Robin): $\alpha(x), \beta(x) \neq 0$ for all x

Boundary Conditions

We write

$$\partial\Omega = S_1 \cup S_2 \cup S_3$$

where

- S_1, S_2, S_3 are pairwise disjoint
- boundary conditions of the kth kind are imposed on S_k.

If any two of these sets are non-empty, we say that we have

mixed boundary conditions.

The Boundary Value Problem

The equations

$$Lu = \varrho F, \qquad Bu = \gamma$$

where

$$\gamma \colon \partial\Omega \to \mathbb{R} \qquad \text{or} \qquad \gamma \colon \partial\Omega \times I \to \mathbb{R}$$

constitute the boundary value problem (L, B) with data $\{\varrho F, \gamma\}$