Spring Term 2017

Vv557 Methods of Applied Mathematics II

Review Questions and Problems

Class Session 1: Distributions

Video Files

06 Further Approaches to the Green Function.mp4
07 Smooth, Compactly Supported Functions.mp4
08 Null Sequences.mp4
09 Test Functions and Distributions.mp4

Review Questions

i) Explain what a test function in $\mathbb{R}^n$ is.

ii) Explain what a null sequence of test functions is and give an example and a counter-example.

iii) Explain what a distribution is.

iv) What is a locally integrable function? Give examples and counter-examples.

v) Define what a regular and a singular distribution is. Give examples.

Exercises

Recall that a sequence (f_n) of functions $f_n \colon I \to \mathbb{C}$, where $I \subset \mathbb{R}$, converges *pointwise* to a function $f \colon I \to \mathbb{C}$ if

$$\lim_{n\to\infty} |f_n(x) - f(x)| = 0 \qquad \text{for all } x \in I.$$

The convergence is *uniform* if

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

Exercise 1.1. Let $\xi \in (0,1) \subset \mathbb{R}$ be fixed. Solve the problem

$$-u'' = f_n(x;\xi), \qquad 0 < x < 1, \qquad\qquad u(0) = u(1) = 0 \tag{1}$$

for

$$f_n(x;\xi) = \begin{cases} n, & |x-\xi| < 1/2n, \\ 0 & \text{otherwise,} \end{cases}$$

with $1/n$ smaller than $\min\{\xi, 1-\xi\}$.

i) For n as above, find the solution u_n of (1). *Solution:*

$$u_n(x) = \begin{cases} (1-\xi)\cdot x & 0 \le x \le \xi - \frac{1}{2n}, \\ (1-\xi)\cdot x - \frac{n}{2}\big(x - \xi + 1/(2n)\big)^2 & \xi - \frac{1}{2n} < x < \xi + \frac{1}{2n}, \\ \xi \cdot (1-x) & \xi + \frac{1}{2n} \le x \le 1. \end{cases}$$

ii) Verify that the sequence of solutions $u_n(x;\xi)$ converges pointwise on $[0,1]$ as $n \to \infty$ to the Green function $g(x,\xi)$ derived in the lecture.

iii) Is the convergence uniform on $[0,1]$? Prove your assertion!

Exercise 1.2. Which of the following are distributions? Justify your response!

i) $T\colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$, $\varphi \mapsto \varphi(-10)$,

ii) $T\colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$, $\varphi \mapsto \varphi(0)^2$,

iii) $T\colon \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}^n$, $\varphi \mapsto \operatorname{grad} \varphi(0)$,

iv) $T\colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$, $\varphi \mapsto \varphi(0) + \varphi(1) + \varphi(2) + \varphi(3) + \dots$,

v) $T_f\colon \mathcal{D} \to \mathbb{R}$, $\varphi \mapsto \int_{\mathbb{R}} f(x)\varphi(x)\, dx$, with

   (a) $f(x) = 1/\sqrt{|x|}$,
   (b) $f(x) = 1/x^2$.

Facultative Exercises

Exercise 1.3. The goal of this exercise is to verify that the eigenfunction expansion of $g(x, \xi)$ coincides with the expression obtained earlier.

i) Use the Fourier series of $(\pi - x)/2$ on the interval $[0, \pi]$ to show that

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}, \qquad 0 \le x \le \pi.$$

ii) Using

$$\sin\alpha \sin\beta = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right), \qquad \alpha, \beta \in \mathbb{R},$$

show that

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)\sin(n\pi\xi)}{n^2} = \begin{cases} (1-\xi)x, & x \le \xi, \\ (1-x)\xi, & x > \xi, \end{cases}$$

for $0 \le x, \xi \le 1$.