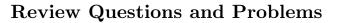
Spring Term 2017

Vv557 Methods of Applied Mathematics II





Class Session 11: Boundary Value Problems for PDEs

Literature

Section 7.1 of E. Zauderer, *Partial Differential Equations of Applied Mathematics*, 3rd Edition, Wiley 2011, Free download from within SJTU network at http://onlinelibrary.wiley.com/book/10.1002/9781118033302.

Video Files

35 Second-Order Equations and Boundary Value Problems.mp4

- 36 The Elliptic Boundary Value Problem.mp4
- 37 The Parabolic Boundary Value Problem.mp4

38 Solution Formula for the Parabolic Boundary Value Problem.mp4

39 Causal Fundamental Solution for the Parabolic Boundary Value Problem.mp4

Review Questions

- i) State the second-order elliptic, hyperbolic and parabolic PDEs as well as the three tyopes of boundary conditions.
- ii) What are mixed boundary conditions?
- iii) How do the concepts of familiar from ODEs carry over to the study of PDEs? Explain the following: formal adjoint, conjunct, Green's formula, adjoint boundary value problem and Green's function.
- iv) What role does the adjoint Green function play in the solution of the parabolic boundary value problem?
- v) How is a causal fundamental solution for a time-dependent PDE defined?

Exercises

Exercise 11.1. Show that the operator on \mathbb{R}^n ,

$$L = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(p(x) \frac{\partial}{\partial x_i} \right) + q(x),$$

is formally self-adjoint.

Exercise 11.2. Derive the solution formula for the elliptic problem (L, B):

$$u(\xi) = \int_{\Omega} g(x,\xi) \varrho(x) F(x) \, dx - \int_{S_1} \frac{p}{\alpha} \gamma \frac{\partial g(\cdot,\xi)}{\partial n} \, d\sigma + \int_{S_2 \cup S_3} \frac{p}{\beta} \gamma g(\cdot,\xi) \, d\sigma$$

It is easiest to consider the cases $\partial \Omega = S_k$, k = 1, 2, 3, separately.

Exercise 11.3. The goal of this exercise is to obtain the d'Alembert solution formula for the Cauchy problem for the wave equation

 $u_{tt} - u_{xx} = F(x, t),$ $x \in \mathbb{R}, t > 0,$ u(x, 0) = f(x), $u_t(x, 0) = h(x),$

from the general solution formula.

i) Let $I = (-L, L) \subset \mathbb{R}$, L > 0, be an interval, let T > 0 be fixed and set $\Omega = I \times (0, T)$. Consider the wave equation problem

$$Lu = u_{tt} - u_{xx} = F \qquad \text{in } \Omega, \qquad \frac{\partial u}{\partial n}\Big|_{\partial I} = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = h(x). \quad (*)$$

Suppose that Green's function $g(x; \xi)$ is known. Write down as explicitly as possible the solution formula for the problem:

- Find L^* and adjoint boundary conditions.
- Find the conjunct of L
- Use Green's formula to find the solution formula.
- ii) Keeping T > 0 fixed, verify that if L is large enough, the fundamental solution $E(x, t; \xi, \tau) = \frac{1}{2}H(t \tau |x \xi|)$ satisfies the boundary conditions of (*).
- iii) Using E and the solution formula obtained in (iii), let $L \to \infty$ to obtain d'Alembert's formula for the solution of the Cauchy problem,

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(y) \, dy + \frac{1}{2} \iint_{\triangle(x,t)} F(y,s) \, dy \, ds \tag{1}$$

where

$$\triangle(x,t) = \{(y,s) \in \mathbb{R}^2 \colon 0 \le s \le t - |x-y|\}.$$

Facultative Exercises

Exercise 11.4. Suppose that $Bu = u|_{\partial\Omega}$ (Dirichlet boundary condition) and let

$$M = \{ u \in C^2(\Omega) \cap C(\overline{\Omega}) \colon Bu = 0 \}.$$

Show that if v satisfies

$$\int_{\partial\Omega} J(u,v) \, d\vec{\sigma} = 0 \qquad \qquad \text{for all } u \in M$$

then

 $v\in M.$

This proves $M^* \subset M$ for the case of Dirichlet boundary conditions.

Exercise 11.5. The same as the previous exercise, but for Neumann and Robin boundary conditions.

Exercise 11.6. Our goal is to find a causal fundamental solution $E(x, t; \xi, \tau)$ for the Cauchy problem of the wave equation,

$$u_{tt} - u_{xx} = F(x, t), \qquad x \in \mathbb{R}, \ t > 0, \qquad u(x, 0) = f(x), \qquad u_t(x, 0) = h(x).$$

i) We consider the Cauchy problem with data $\{\delta(x-\xi)\delta(t-\tau); 0, 0\}$. Take the Fourier transform with respect to the x-variable to obtain the ODE

$$\widehat{E}_{tt} + k^2 \widehat{E} = \delta(t - \tau) e^{ik\xi}$$

for $\widehat{E}(k,t;\xi,\tau)$.

ii) Solve the ODE with suitable initial conditions to find

$$\widehat{E}(k,t;\xi,\tau) = H(t-\tau)\frac{\sin(k(t-\tau))}{k}e^{ik\xi}.$$

iii) Calculate the inverse Fourier transform and obtain

$$E(x,t;\xi,\tau) = \frac{1}{2}H(t-\tau - |x-\xi|) = \begin{cases} 1/2, & \xi - t + \tau < x < \xi + t - \tau, \\ 0, & \text{otherwise.} \end{cases}$$

If necessary, you may use without proof that $\int_0^\infty \frac{\sin(ka)}{k} dk = \pi/2$ for a > 0.

iv) Verify that $E(x,t;\xi,\tau)$ is a fundamental solution by inserting it into the wave equation and calculating the derivatives in the distributional sense. (This is a fiddly, but elementary, calculation and a good opportunity to review distributional derivatives.)