Spring Term 2017

Vv557 Methods of Applied Mathematics II

Review Questions and Problems

JOINT INSTITUTE
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Class Session 11: Boundary Value Problems for PDEs

Literature

Section 7.1 of E. Zauderer, *Partial Differential Equations of Applied Mathematics*, 3rd Edition, Wiley 2011, Free download from within SJTU network at http://onlinelibrary.wiley.com/book/10.1002/9781118033302.

Video Files

35 Second-Order Equations and Boundary Value Problems.mp4
36 The Elliptic Boundary Value Problem.mp4
37 The Parabolic Boundary Value Problem.mp4
38 Solution Formula for the Parabolic Boundary Value Problem.mp4
39 Causal Fundamental Solution for the Parabolic Boundary Value Problem.mp4

Review Questions

i) State the second-order elliptic, hyperbolic and parabolic PDEs as well as the three tyopes of boundary conditions.

ii) What are mixed boundary conditions?

iii) How do the concepts of familiar from ODEs carry over to the study of PDEs? Explain the following: formal adjoint, conjunct, Green's formula, adjoint boundary value problem and Green's function.

iv) What role does the adjoint Green function play in the solution of the parabolic boundary value problem?

v) How is a causal fundamental solution for a time-dependent PDE defined?

Exercises

Exercise 11.1. Show that the operator on $\mathbb{R}^n$,

$$L = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(p(x)\frac{\partial}{\partial x_i}\right) + q(x),$$

is formally self-adjoint.

Exercise 11.2. Derive the solution formula for the elliptic problem (L, B):

$$u(\xi) = \int_{\Omega} g(x,\xi)\varrho(x)F(x)\,dx - \int_{S_1} \frac{p}{\alpha}\gamma\frac{\partial g(\,\cdot\,,\xi)}{\partial n}\,d\sigma + \int_{S_2\cup S_3} \frac{p}{\beta}\gamma g(\,\cdot\,,\xi)\,d\sigma$$

It is easiest to consider the cases $\partial\Omega = S_k$, $k = 1, 2, 3$, separately.

Exercise 11.3. The goal of this exercise is to obtain the d'Alembert solution formula for the Cauchy problem for the wave equation

$$u_{tt} - u_{xx} = F(x,t), \qquad x \in \mathbb{R},\ t > 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = h(x),$$

from the general solution formula.

i) Let $I = (-L, L) \subset \mathbb{R}$, $L > 0$, be an interval, let $T > 0$ be fixed and set $\Omega = I \times (0, T)$. Consider the wave equation problem

$$Lu = u_{tt} - u_{xx} = F \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n}\Big|_{\partial I} = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = h(x). \qquad (*)$$

Suppose that Green's function $g(x;\xi)$ is known. Write down as explicitly as possible the solution formula for the problem:

- Find L^* and adjoint boundary conditions.
- Find the conjunct of L
- Use Green's formula to find the solution formula.

ii) Keeping $T > 0$ fixed, verify that if L is large enough, the fundamental solution $E(x,t;\xi,\tau) = \frac{1}{2}H(t - \tau - |x - \xi|)$ satisfies the boundary conditions of $(*)$.

iii) Using E and the solution formula obtained in (iii), let $L \to \infty$ to obtain d'Alembert's formula for the solution of the Cauchy problem,

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy + \frac{1}{2}\iint_{\triangle(x,t)} F(y,s)\,dy\,ds \qquad (1)$$

where

$$\triangle(x,t) = \{(y,s) \in \mathbb{R}^2 \colon 0 \le s \le t - |x-y|\}.$$

Facultative Exercises

Exercise 11.4. Suppose that $Bu = u|_{\partial\Omega}$ (Dirichlet boundary condition) and let

$$M = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) \colon Bu = 0\}.$$

Show that if v satisfies

$$\int_{\partial\Omega} J(u,v)\,d\vec{\sigma} = 0 \qquad \text{for all } u \in M$$

then

$$v \in M.$$

This proves $M^* \subset M$ for the case of Dirichlet boundary conditions.

Exercise 11.5. The same as the previous exercise, but for Neumann and Robin boundary conditions.

Exercise 11.6. Our goal is to find a causal fundamental solution $E(x,t;\xi,\tau)$ for the Cauchy problem of the wave equation,

$$u_{tt} - u_{xx} = F(x,t), \qquad x \in \mathbb{R},\ t > 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = h(x).$$

i) We consider the Cauchy problem with data $\{\delta(x-\xi)\delta(t-\tau); 0, 0\}$. Take the Fourier transform with respect to the x-variable to obtain the ODE

$$\widehat{E}_{tt} + k^2\widehat{E} = \delta(t-\tau)e^{ik\xi}$$

for $\widehat{E}(k,t;\xi,\tau)$.

ii) Solve the ODE with suitable initial conditions to find

$$\widehat{E}(k,t;\xi,\tau) = H(t-\tau)\frac{\sin(k(t-\tau))}{k}e^{ik\xi}.$$

iii) Calculate the inverse Fourier transform and obtain

$$E(x,t;\xi,\tau) = \frac{1}{2}H(t-\tau-|x-\xi|) = \begin{cases} 1/2, & \xi - t + \tau < x < \xi + t - \tau, \\ 0, & \text{otherwise.} \end{cases}$$

If necessary, you may use without proof that $\int_0^\infty \frac{\sin(ka)}{k}\,dk = \pi/2$ for $a > 0$.

iv) Verify that $E(x,t;\xi,\tau)$ is a fundamental solution by inserting it into the wave equation and calculating the derivatives in the distributional sense. (This is a fiddly, but elementary, calculation and a good opportunity to review distributional derivatives.)