Spring Term 2017

Vv557 Methods of Applied Mathematics II

Review Questions and Problems

JOINT INSTITUTE
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Class Session 12: Eigenfunction Expansions

Literature

Section 7.3 of E. Zauderer, *Partial Differential Equations of Applied Mathematics*, 3rd Edition, Wiley 2011, Free download from within SJTU network at http://onlinelibrary.wiley.com/book/10.1002/9780470906538.

Video Files

40 The Eigenvalue problem for the Elliptic Operator.mp4
41 Full Eigenfunction Expansions.mp4
42 Partial Eigenfunction Expansions.mp4

Review Questions

i) Summarize the properties of the eigenvalues of the elliptic operator.

ii) Explain what full and partial eigenfunction expansions are and the difference between them.

Exercises

Exercise 11.1. The goal of this exercise is to find the full eigenfunction expansion of Green's function for Dirichlet problem for the Helmholtz equation on the square

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon 0 < x_1 < 1,\ 0 < x_2 < 1\}.$$

Green's function satisfies

$$Lg = -\Delta g(x,\xi) + k^2 g(x,\xi) = \delta(x,\xi), \qquad x,\xi \in \operatorname{int}\Omega, \qquad\qquad g(\,\cdot\,,\xi)\big|_{\partial\Omega} = 0.$$

where $k^2 > 0$ is a parameter.

Find the full eigenfunction expansion for g. For which values of k^2 does this eigenfunction expansion exist?

Exercise 11.2. The goal of this exercise is to find a partial eigenfunction expansion for Green's function for the Dirichlet problem for the half-disk

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon |x|^2 \le 1,\ x_2 \ge 0\}.$$

Green's function satisfies

$$-\Delta g(x,\xi) = \delta(x,\xi), \qquad x,\xi \in \operatorname{int}\Omega, \qquad\qquad g(\,\cdot\,,\xi)\big|_{\partial\Omega} = 0.$$

For convenience, we introduce polar coordinates.

i) Separate variables using the Laplace operator

$$\Delta_{(r,\theta)} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

and find the angular eigenfunctions and the eigenvalues.

ii) Give the formal partial eigenfunction expansion for Green's function in terms of suitable θ eigenfunctions (use the boundary conditions for g). Do not yet determine the coefficient functions.

iii) Determine the one-dimensional Green's function problem that the coefficients must satisfy.

iv) Solve the Green's function problem and find the coefficients, giving the partial eigenfunction expansion.

The following Exercises may be completed by Thursday:

Exercise 11.3. Following the discussion on pages 456-457 of *Zauderer*, find a partial eigenfunction expansion of a causal Green function for the wave equation

$$\Delta u = \frac{\partial^2 u}{\partial t^2}, \qquad x \in \Omega, \quad t > 0$$

on the square

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon 0 < x_1 < 1,\ 0 < x_2 < 1\}.$$

with Dirichlet boundary conditions on $\partial\Omega$. Give the expansion for the direct Green function g as well as for the adjoint function g^*.

Exercise 11.4. The goal of this exercise is to find a partial eigenfunction expansion for Green's function for the Dirichlet problem for the half-strip

$$S_a^+ = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon 0 < x_2 < a,\ x_1 > 0\}.$$

Green's function satisfies

$$-\Delta g(x, \xi) = \delta(x, \xi), \qquad x, \xi \in S_a^+, \qquad g(\,\cdot\,, \xi)\big|_{\partial S_a^+} = 0.$$

i) Separate variables in the Laplace equation

$$\Delta u = 0$$

and find the x_2 eigenfunctions and the eigenvalues.

ii) Give the formal partial eigenfunction expansion for Green's function in terms of the x_2 eigenfunctions (use the boundary conditions for g). Do not yet determine the coefficient functions.

iii) Determine the one-dimensional Green's function problem that the coefficients must satisfy.

iv) Solve the Green's function problem and find the coefficients, giving the partial eigenfunction expansion.

Remark: You should try to solve this exercise yourself. However, the solution is also published in Maejo Int. J. Sci. Technol. 2010, 4(01), 88-92 - it seems as if certain journals will publish even results of homework exercises!)