Spring Term 2017

Vv557 Methods of Applied Mathematics II

Review Questions and Problems

JOINT INSTITUTE
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Class Session 13: Introduction to the Method of Images

Literature

Section 7.5 of E. Zauderer, *Partial Differential Equations of Applied Mathematics*, 3rd Edition, Wiley 2011, Free download from within SJTU network at http://onlinelibrary.wiley.com/book/10.1002/9780470906538.

Video Files

43 The Method of Images.mp4
44 Exploiting Symmetries for Image Charges.mp4

Exercises

Exercise 13.1. Use Green's function for the Dirichlet problem on the upper half-plane $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ to show that the solution to

$$\Delta u = 0 \qquad \text{on } \mathbb{H}, \qquad\qquad u = h \qquad \text{on } \partial\mathbb{H}$$

is given by

$$u(x_1, x_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} h(y) \frac{x_2}{x_2^2 + (x_1 - y)^2}\, dy.$$

Exercise 13.2.

i) Use the method of images to find Green's function for the Neumann problem for the Laplace operator on the upper half-plane, i.e., $\Omega = \{x \in \mathbb{R}^2 : x_2 > 0\}$.

ii) Give the formula for the general solution for the problem

$$\Delta u = 0 \qquad \text{on } \Omega = \{x \in \mathbb{R}^2 : x_2 > 0\}, \qquad\qquad \frac{\partial u}{\partial n}\Big|_{\Omega} = -\frac{\partial u}{\partial x_2}\Big|_{x_2=0} = f(x_1).$$

Find the solution if $f(x_1) = 1$.

iii) Show that the method of images fails if Ω is replaced by the unit disk with Neumann boundary conditions.

Exercise 13.3. Find Green's function for the Dirichlet problem on the first quadrant $\Omega = \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\big\}$ and give a solution formula for the Dirichlet problem

$$\Delta u = \varrho \qquad \text{on } \Omega, \qquad\qquad u = f \qquad \text{on } \partial\Omega$$

Exercise 13.4. In this exercise[1] the method of images is used to find Green's function $g(x, \xi)$ for the Dirichlet problem on the wedge

$$\Omega = \{x \in \mathbb{R}^2 \colon x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ r > 0,\ 0 < \theta < \pi/3\}.$$

i) You will need five image charges, $\xi^{(1)}, \ldots, \xi^{(5)}$. Sketch their location and give their position in polar coordinates $(r^{(k)}, \theta^{(k)})$, $k = 1, \ldots, 5$, supposing that ξ is located at (r_0, θ_0). Write down the resulting Green's function as a sum of fundamental solutions.

   Hint: You may find it useful to introduce the complex variable $z = r\exp(i\theta)$

ii) If ξ is located at $(r, \theta) = (1, \pi/6)$, show that Green's function in polar coordinates is given by

$$g(r, \theta; 1, \pi/6) = -\frac{1}{4\pi}\ln\left(\frac{r^6 - 2r^3\sin(3\theta) + 1}{r^6 + 2r^3\sin(3\theta) + 1}\right)$$

iii) Give a solution formula for the Dirichlet problem

$$\Delta u = 0 \qquad \text{on } \Omega, \qquad u(r, \theta = 0) = f_1(r), \qquad u(r, \theta = \pi/3) = f_2(r), \qquad r > 0.$$

Facultative Exercises

Exercise 13.5. Consider again the first quadrant $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 > 0,\ x_2 > 0\}$. Use the method of images to find Green's function for the mixed problem

$$\Delta g(x; \xi) = \delta(x - \xi) \quad \text{on } \Omega, \qquad g(x_1, 0; \xi) = f(x_1), \quad x_1 > 0, \qquad \left.\frac{\partial g(x; \xi)}{\partial x_1}\right|_{x_1 = 0} = h(x_2), \quad x_2 > 0.$$

Give a solution formula for the problem

$$\Delta u = \varrho \quad \text{on } \Omega, \qquad u(x_1, 0) = f(x_1), \quad x_1 > 0, \qquad \left.\frac{\partial u(x)}{\partial x_1}\right|_{x_1 = 0} = h(x_2), \quad x_2 > 0.$$

Exercise 13.6.

i) Use Green's function for the Dirichlet problem on the unit disc $\mathbb{D} = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < 1\}$ to show that the solution to

$$\Delta u = 0 \qquad \text{on } \mathbb{D}, \qquad u = h \qquad \text{on } \partial\mathbb{D}$$

   is given (in polar coordinates) by

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \varphi)} h(\varphi)\, d\varphi.$$

   This is known as the *Poisson integral formula.*

ii) Find the solution of the Dirichlet problem

$$\Delta_{r,\theta} u(r, \theta) = 0, \qquad (r, \theta) \in (0, 1) \times [-\pi, \pi), \qquad u(1, \varphi) = \begin{cases} 2 & -\pi \le \theta < -\pi/2, \\ 1 & -\pi/2 \le \theta < \pi/2, \\ 0 & \pi/2 \le \theta < \pi, \end{cases}$$

   in terms of elementary functions. Plot the graph of the solution using a computer algebra system. You may use that

$$\frac{1}{2\pi}\int \frac{1 - r^2}{1 + r^2 - 2r\cos(t - \varphi)}\, dt = \frac{1}{\pi}\arctan\Big(\frac{1 + r}{1 - r}\tan\frac{t - \varphi}{2}\Big).$$

   (Pay careful attention the branches of the arctangent. The solution will be a continuous function of φ.)

[1] Taken from the Natural Sciences Tripos examination, Cambridge University, 2008