Spring Term 2017

Vv557 Methods of Applied Mathematics II

Review Questions and Problems

Class Session 2: Operations on Distributions

Video Files

10 Elementary Operations on Distributions.mp4
11 The Weak Derivative.mp4
12 Two Applications of the Weak Derivative.mp4

Review Questions

i) Explain how operations for functions are extended to distributions "by duality".

ii) What are Green's first and second identities?

iii) Why is the function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = 1/x$ not a distribution? Explain what the principal value of g is.

Exercises

Exercise 2.1. Let $\xi \in (0,1)$ be fixed. The goal of this exercise is to show that the Green's function $g(x,\xi)$ (introduced and defined in the lecture video) for the problem

$$-u'' = f(x), \qquad 0 < x < 1, \qquad\qquad u(0) = u(1) = 0$$

satisfies

$$-g'' = \delta(x - \xi), \qquad 0 < x < 1, \tag{1}$$

in the distributional sense. This will require the definition of distributions on the open set $\Omega = (0,1) \subset \mathbb{R}$.

Proceed as follows: Define first $\mathcal{D}(0,1) := \{\varphi \in \mathcal{D}(\mathbb{R})\colon \operatorname{supp}\varphi \subset (0,1)\}$ and then $\mathcal{D}'(0,1)$ as the set of continuous linear functionals on $\mathcal{D}(0,1)$. Regard $g(\,\cdot\,,\xi)$ as an element of $\mathcal{D}'(0,1)$ and differentiate it as a distribution. Note that the test functions will have compact support in the interval $(0,1) \subset \mathbb{R}$.

Exercise 2.2.

i) Verify that the Cauchy principal value $\mathcal{P}(1/x)$ defines a distribution, i.e., that it is a continuous linear functional on $\mathcal{D}(\mathbb{R})$.

ii) Verify that $x\mathcal{P}(1/x) = 1$ in the sense of distributions.

Exercise 2.3. While $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 1/x$ is not a distribution, the function $h\colon \mathbb{R}^3 \to \mathbb{R}$, $h(x) = 1/|x|$ is locally integrable and hence a regular distribution. Verify this!

Facultative Exercises

Exercise 2.4. Show that

$$g \in \mathcal{D}'(\mathbb{R}^2), \qquad\qquad g(x) = -\frac{1}{2\pi}\log|x|$$

satisfies $\Delta g = \delta(x)$ in the distributional sense.

Exercise 2.5. For $u\colon \mathbb{R}^2 \to \mathbb{R}$ given by[1]

$$u(x,t) = \begin{cases} 1/2 & t - |x| > 0, \\ 0 & \text{otherwise} \end{cases}$$

calculate $u_{tt} - u_{xx}$.

Exercise 2.6. Prove the following statement: Let $I \subset \mathbb{R}$ be an open interval and $f \in L^1_{\text{loc}}(I) \cap C^1(I \setminus \{\xi\})$ such that the left- and right-hand limits of f and f' at ξ exist. Denote

$$[f]_\xi := \lim_{\varepsilon \to 0} \big(f(\xi + \varepsilon) - f(\xi - \varepsilon) \big).$$

Then

$$(T_f)' = [f]_\xi \cdot \delta(x - \xi) + T_{f'}$$

where we define $f'(\xi)$ to have any value we like.

Exercise 2.7. Show that

$$\frac{d}{dx} \mathcal{P}\left(\frac{1}{x}\right) = -\mathcal{P}\left(\frac{1}{x^2}\right),$$

where

$$\mathcal{P}\left(\frac{1}{x^2}\right)(\varphi) := \lim_{\varepsilon \searrow 0} \int_{|x|>\varepsilon} \frac{1}{x^2} \big(\varphi(x) - \varphi(0)\big)\, dx.$$

(5 Marks)

[1] Zuily, C., *Problems in Distributions and Partial Differential Equations*, Exercise 28