# Spring Term 2017

# Vv557 Methods of Applied Mathematics II

# Review Questions and Problems

## Class Session 3: Families of Distributions

## Video Files

13 Families of Distributions.mp4
14 Delta Families.mp4

## Review Questions

i) How is convergence of a family of distributions defined?

ii) What is a delta sequence or a delta family? Give examples!

## Exercises

**Exercise 3.1.** For fixed $\alpha > 0$, consider the sequence $(f_k)$ of continuous functions[1]

$$f_k \colon \mathbb{R} \to \mathbb{R}, \qquad\qquad f_k(x) = k^\alpha H(x) x e^{-kx}$$

Show that

i) $f_k(x) \to 0$ as $k \to \infty$ for all $x \in \mathbb{R}$ and any value of $\alpha > 0$,

ii) $f_k \to 0$ uniformly on $\mathbb{R}$ as $k \to \infty$ if $\alpha < 1$,

iii) $\int_{\mathbb{R}} |f_k(x)|\, dx \to 0$ as $k \to \infty$ if $\alpha < 2$,

iv) $T_{f_k} \to 0$ as $k \to \infty$ in the sense of distributions if $\alpha < 2$,

v) $T_{f_k} \to T_\delta$ as $k \to \infty$ in the sense of distributions if $\alpha = 2$,

vi) $T_{f_k}$ does not converge as $k \to \infty$ in the sense of distributions if $\alpha > 2$.

**Exercise 3.2.** For $z = re^{i\varphi} \in \mathbb{C}$ define $\arg z := \varphi$. For $\varepsilon > 0$ and $x \in \mathbb{R}$ define[2]

$$\ln(x + i\varepsilon) := \ln(|x + i\varepsilon|) + i \arg(x + i\varepsilon).$$

i) Define $f_\varepsilon \colon \mathbb{R} \to \mathbb{C}$, $f_\varepsilon(x) = \ln(x + i\varepsilon)$ for $\varepsilon > 0$. Let

$$f_0 \colon \mathbb{R} \to \mathbb{C}, \qquad\qquad f_0(x) = \begin{cases} \ln(x) & x > 0, \\ \ln(|x|) + i\pi & x < 0, \\ 0 & x = 0. \end{cases}$$

Show that

$$\lim_{\varepsilon \searrow 0} f_\varepsilon = f$$

in the sense of distributions (i.e., $T_{f_\varepsilon} \to T_{f_0}$ in $\mathcal{D}'$).

ii) Calculate the derivative of $f_0$ as a distribution. Recall that the derivative of $\ln(|x|)$ was the principal value of $1/x$, but $f_0$ also has a jump discontinuity that needs to be taken into account.

[1] *Stakgold*, Ex. 2.2.3

[2] Zuily, C., *Problems in Distributions and Partial Differential Equations*, Exercise 47

iii) Deduce that

$$\lim_{\varepsilon \searrow 0} \frac{1}{x + i\varepsilon} = -i\pi\delta(x) + \mathcal{P}\left(\frac{1}{x}\right)$$

in the sense of distributions.

iv) Show similarly that

$$\lim_{\varepsilon \searrow 0} \frac{1}{x - i\varepsilon} = i\pi\delta(x) + \mathcal{P}\left(\frac{1}{x}\right)$$

and conclude that

$$\lim_{\varepsilon \searrow 0} \frac{1}{\pi(x^2 + \varepsilon^2)} = \delta(x).$$

# Facultative Exercises

**Exercise 3.3.** Let $\{f_\alpha\}$ be a family of nonnegative locally integrable functions on $\mathbb{R}^n$. Make the following assumptions:[3]

(A) For some $R > 0$, $\lim_{\alpha \to \alpha_0} \int_{|x|<R} f_\alpha(x)\, dx = 1$,

(B) For every $R > 0$, $f_\alpha(x) \to 0$ as $\alpha \to \alpha_0$, uniformly for $|x| > R$.

Show the following:

i) If (A) holds for some $R > 0$, then (A) also holds for any $R > 0$.

ii) Suppose that $\varphi$ is any function which is continuous at $x = 0$ and that satisfies $\int_{\mathbb{R}^n} |\varphi(x)|\, dx < \infty$. Then

$$\lim_{\alpha \to \alpha_0} \int_{\mathbb{R}^n} f_\alpha(x)\varphi(x)\, dx = \varphi(0).$$

Clearly, this shows that $\{f_\alpha\}$ is a delta family as $\alpha \to \alpha_0$. However, since very few assumptions on $\varphi$ are made, this result can be applied even more generally.

*Hint:* Fix a suitable value of $R > 0$ and estimate the terms in

$$\int_{\mathbb{R}^n} f_\alpha(x)\varphi(x)\, dx - \varphi(0) = \int_{|x|\le R} f_\alpha(x)(\varphi(x) - \varphi(0)\, dx + \varphi(0)\left(\int_{|x|\le R} f_\alpha(x)\, dx - 1\right) + \int_{|x|\ge R} f_\alpha(x)\varphi(x)\, dx$$

iii) Consider a complex number $z = re^{i\theta}$ and use the geometric series formula to show that

$$1 + 2\sum_{n=1}^{\infty} z^n = 1 + \frac{2z}{1-z} = \frac{1 - r^2 - 2ir\sin\theta}{1 - 2r\cos\theta + r^2}$$

From this and $\operatorname{Re} z^n = r^n \cos(n\theta)$ , deduce that

$$\frac{1 - r^2}{1 + r^2 - 2r\cos\theta} = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta)$$

Integrate to show that

$$\int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}\, d\theta = 2\pi.$$

iv) Combine the previous results to deduce that the family

$$f_r(\theta) = \begin{cases} \frac{1}{2\pi} \cdot \frac{1-r^2}{1+r^2-2r\cos\theta} & |\theta| \le \pi, \\ 0 & |\theta| > \pi, \end{cases} \qquad 0 \le r < 1,$$

converges to $\delta(\theta)$ as $r \nearrow 1$.

[3] *Stakgold*, Ex. 2.2.4, 2.2.9