Spring Term 2017

Vv557 Methods of Applied Mathematics II

Review Questions and Problems

Class Session 5: The Fourier Transform for Tempered Distributions

Video Files

18 Tempered Distributions.mp4
19 Application of the Fourier Transform to Partial Differential Equations.mp4

Review Questions

i) What is a tempered distribution?

ii) How is the Fourier transform defined for tempered distributions?

iii) Explain how the convolution can be defined for tempered distributions.

Exercises

Exercise 5.1. Calculate the Fourier transforms of the following elements in $\mathcal{S}'(\mathbb{R})$:

i) $\begin{cases} e^{-\varepsilon x} & x \geq 1, \\ 0 & x < 1, \end{cases} \qquad \varepsilon > 0,$

ii) $\sin(3x - 2)$,

iii) $x^2 \cos(x)$,

iv) $xH(x-2)$,

v) $x^2\delta(x-1)$.

Exercise 5.2. A distribution $T \in \mathcal{D}'$ is said to be even if $T\varphi = T\widetilde{\varphi}$ for all test functions φ, where $\widetilde{\varphi}(x) = \varphi(-x)$. The distribution is said to be odd if $T\varphi = -T\widetilde{\varphi}$.

Show that if $T \in \mathcal{S}'$ is even (odd), then the Fourier transform $\widehat{T} \in \mathcal{S}'$ is even (odd).

Exercise 5.3. Consider the wave equation problem for a function $u\colon \mathbb{R}^2 \to \mathbb{R}$,

$$u_{tt} - u_{xx} = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

Take the Fourier transform of the equation with respect to the x-variable to obtain an ODE in the t-variable and solve the ODE to obtain

$$\widehat{u}(\xi,t) = \hat{f}(\xi)\cos(\xi t) + \frac{\hat{g}(\xi)}{\xi}\sin(\xi t).$$

Then calculate the inverse Fourier transform (in the distributional sense) to obtain a solution formula for $u(x,t)$.