Spring Term 2017

Vv557 Methods of Applied Mathematics II

Review Questions and Problems

Class Session 7: Causal Fundamental Solutions and Initial Value Problems

Video Files

22 Fundamental Solutions.mp4
23 Initial Value Problems, Independence and the Wronksian.mp4
24 The Homogeneous Equation with Non-Vanishing Initial Conditions.mp4
25 The Inhomogeneous Equation.mp4

Review Questions

i) What is a fundamental solution to a differential equation? Are fundamental solutions unique?

ii) What is a causal fundamental solution to a differential equation that involves a time variable?

iii) How is a causal fundamental solution found?

iv) What is a system of independent solutions to an ODE?

v) How is the Wronksian defined?

vi) What is the relationship between the Wronksian and linear dependence/independence of arbitrary functions? Of solutions to an ODE?

vii) What is the solution formula for the initial value problem for an inhomogeneous ODE?

Exercises

Exercise 6.1. We want to find a fundamental solution of the stationary equation for a simply supported beam, i.e., a function $g(x, \xi)$ satisfying

$$\frac{d^4 g}{dx^4} = \delta(x - \xi), \qquad\qquad 0 < x, \xi < 1,$$

with boundary conditions

$$g(0, \xi) = g''(0, \xi) = g(1, \xi) = g''(1, \xi) = 0.$$

i) Find a causal fundamental solution, i.e., a function E satisfying

$$\frac{d^4 E}{dx^4} = \delta(x - \xi), \qquad\qquad 0 < x, \xi < 1,$$

and $E(x) = 0$ for $x < \xi$.

ii) Add a solution of the homogeneous equation $\frac{d^4 u}{dx^4} = 0$ to E to obtain a function that satisfies the boundary conditions.

Exercise 6.2. Use the solution formula of Video 25 for an inhomogeneous ODE to find the general solution of

$$y'' - 4y' + 4y = te^{2t},$$

giving also a fundamental system of solutions for the homogeneous equation.

Facultative Exercises

Exercise 6.3. We want to find a fundamental solution of the stationary equation for a travelling wave with wavenumber k, i.e., a function $g(x, \xi)$ satisfying

$$-\frac{d^2 g}{dx^2} - k^2 g = \delta(x - \xi), \qquad 0 < x, \xi < 1,$$

with boundary conditions

$$g(0, \xi) = g(1, \xi) = 0.$$

i) Find a causal fundamental solution, i.e., a function E satisfying

$$-\frac{d^2 E}{dx^2} - k^2 E = \delta(x - \xi), \qquad 0 < x, \xi < 1,$$

and $E(x) = 0$ for $x < \xi$.

ii) Add a solution of the homogeneous equation $-\frac{d^2 u}{dx^2} - k^2 u = 0$ to E to obtain a function that satisfies the boundary conditions.

iii) Another approach to the same problem: Use the Fourier transform to find a fundamental solution on $\mathbb{R}$, i.e., a function E satisfying

$$-\frac{d^2 E}{dx^2} - k^2 E = \delta(x - \xi), \qquad x, \xi \in \mathbb{R}.$$

iv) Add a solution of the homogeneous equation $-\frac{d^2 u}{dx^2} - k^2 u = 0$ to E to obtain a function that satisfies the boundary conditions.

Exercise 6.4. Prove Abel's formula for the Wronksian (see Video 24) using, for instance, the definition of the determinant by Leibniz's formula (or any other method).

Exercise 6.5. The causal fundamental solution $E(t, \tau)$ is known as the *impulse response* in electrical engineering. E satisfies

$$LE = \delta(t - \tau), \qquad E(t, \tau) = 0 \quad \text{for } t < \tau.$$

The *step response* $F(t, \tau)$ is the solution of

$$LF = H(t - \tau), \qquad F(t, \tau) = 0 \quad \text{for } t < \tau.$$

i) Show that —$\partial F/\partial \tau$ is a fundamental solution. Since $\partial F/\partial \tau = 0$ for $t < \tau$, we must have $-\partial F/\partial \tau = E(t, \tau)$.

ii) Show that the solution of

$$Lu = f(t), \qquad u(a) = u'(a) = \cdots = u^{(p-1)} = 0$$

can be written in either of the forms

$$u(t) = \int_a^t E(t, \tau) f(\tau)\, d\tau,$$
$$u(t) = F(t, a) f(a) + \int_a^t F(t, \tau) f'(\tau)\, d\tau$$

iii) What simplifications are possible when the coefficients in L are constant?