UNIVERSITY OF MICHIGAN

UM-SJTU JOINT INSTITUTE
上海交通大学交大密西根
联合学院

SHANGHAI JIAO TONG UNIVERSITY

Applied Calculus III

Exercise Set 10

Date Due: 12:00 PM, Thursday, the 29th of July 2010

Office hours: Tuesdays, 12:00-1:00 PM and on the SAKAI system

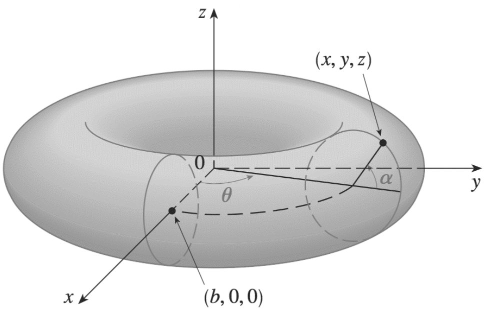
Exercise 1. For each of the following surfaces in $\mathbb{R}^3$ a parametrization of the form $\varphi(u,v)$ with $\varphi\colon \Omega \to \mathbb{R}^3$ a suitable domain $\Omega \subset \mathbb{R}^2$ is given. Calculate the area of each surface, and sketch the surface.

i) **Cone**: Let $\alpha \in (0,\pi)$ be fixed, $\Omega = [0,2\pi] \times [0,h]$ and $\varphi(\theta,z) = (z\sin\alpha\cos\theta, z\sin\alpha\sin\theta, z\cos\alpha)$.

ii) **Paraboloid**: Let $a > 0$ be fixed, $\Omega = [0,2\pi] \times [0,h]$ and $\varphi(\theta,t) = (at\cos\theta, at\sin\theta, t^2)$.

iii) **Screw**: Let $\Omega = [0,1] \times [0,2\pi]$ and $\varphi(r,\phi) = (r\cos\phi, r\sin\phi, \phi)$. Compare the surface area of the screw with that of the unit circle (the unit circle is the screw's projection onto the x_1-x_2-plane.)

$(2+2+(2+1)$ Marks)



Exercise 2. Find a parametric representation for the torus obtained by rotating about the z-axis the circle in the $x-z$-plane with center $(b,0,0)$ and radius $a < b$. *Hint:* Take as parameters the angles θ and α shown in the figure.

Then find the surface area of the torus.
$(2+2$ Marks)

Exercise 3. Find the area of each of the following surfaces in $\mathbb{R}^3_{(x,y,z)}$:

i) The part of the plane $x + 2y + z = 4$ that lies inside the cylinder $x^2 + y^2 = 9$,

ii) The part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0,0)$, $(0,1)$ and $(2,1)$,

iii) The part of the cylinder $x^2 + z^2 = a^2$ that lies in the cylinder $x^2 + y^2 = a^2$

$(3 \times 2$ Marks)

Exercise 4. For each of the following surfaces in $\mathbb{R}^3$ a parametrization of the form $\varphi(u,v)$ with $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ is given. Find an equation of the tangent plane to the given surface at the specified point $p \in \mathbb{R}^3$. Use a computer to graph the surface together with the tangent plane.

i) $\varphi(u,v) = (u+v, 3u^2, u-v)$, $p = (2,3,0)$,

ii) $\varphi(u,v) = (u^2, v^2, uv)$, $p = \varphi(1,1)$,

iii) $\varphi(u,v) = (uv, u\sin v, v\cos u)$, $p = \varphi(0,\pi)$.

$(3 \times (2+1)$ Marks)

Exercise 5. Evaluate the following surface integrals:

i) $\iint_{\mathcal{S}} y \, d\sigma$ where $\mathcal{S}$ is the surface given by $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leq x \leq 1$, $0 \leq y \leq 1$.

ii) $\iint_{\mathcal{S}} y \, d\sigma$ where $\mathcal{S}$ is the part of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + y^2 = 9$.

iii) $\iint_{\mathcal{S}} (x^2 + y^2) z \, d\sigma$ where $\mathcal{S}$ is the hemisphere given by $x^2 + y^2 + z^2 = 4$, $z > 0$.

(3 × 2 Marks)

Exercise 6. The electrostatic potential $V(p)$ at a point $p \in \mathbb{R}^3$ induced by a charged surface $\mathcal{S}$ is given by

$$V(p) = \frac{1}{4\pi\varepsilon_0} \iint\limits_{\mathcal{S}} \frac{\varrho(\,\cdot\,)}{\mathrm{dist}(p,\,\cdot\,)} \, d\sigma$$

where ϱ is the charge density of the surface. Let $\mathcal{S}$ be a uniformly charged circular disk of radius b (and negligible thickness) carrying a total charge Q.

i) Find the electric field $E = -\nabla V$ at a point p on the axis through the center of and perpendicular to the disk.

ii) Let R denote the distance of p from the disk. By expanding E in terms of b/R show that if $R \gg b$ the electric field strength approaches that of a point charge situated at the center of the disk.

(2 + 2 Marks)

Exercise 7. A charge Q is distributed uniformly over the wall fo a circular tube of radius b and height h. Determine V and E on its axis

i) at a point outside the tube, then

ii) at a point inside the tube.

(2 + 2 Marks)