# Applied Calculus III

## Exercise Set 11

**Date Due: 12:00 PM, Tuesday, the 10th of August 2010**

**Office hours: Tuesdays, 12:00-1:00 PM and on the SAKAI system**

**Exercise 1.** Let $A \in \mathrm{Mat}(2 \times 2, \mathbb{R})$ be symmetric, i.e.,

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Let $\Delta = \det A$. Prove that

i) $A$ positive definite $\Leftrightarrow a > 0$ and $\Delta > 0$

ii) $A$ negative definite $\Leftrightarrow a < 0$ and $\Delta > 0$

iii) $A$ indefinite $\Leftrightarrow \Delta < 0$

**(3 Marks)**

**Exercise 2.** Find all local and global extrema (if they exist) of the following real functions on their domains.

i) $\operatorname{dom} f = \mathbb{R}^2$, $f(x, y) = x^2 + xy + y^2 + x + y + 1$

ii) $\operatorname{dom} f = \mathbb{R}^2 \setminus \{0\}$, $f(x, y) = 1/y - 1/x - 4x + y$

iii) $\operatorname{dom} f = \mathbb{R}^2$, $f(x, y) = \sqrt{x^2 + y^2}$

iv) $\operatorname{dom} f = \{(x, y) \in \mathbb{R}^2 : 0 \le x, y \le \pi/2\}$, $f(x, y) = \sin x + \sin y + \sin(x + y)$

**(2 + 2 + 1 + 4 Marks)**

**Exercise 3.** The effect $E(x, t)$ of $x$ units of some medicine $t$ hours after ingestion is frequently modeled by

$$E(x, t) = x^2(a - x)t^2e^{-t}, \qquad\qquad 0 \le x \le a, \quad t \ge 0.$$

Find the dosage $x$ and the time $t$ such that the effect is maximal.
**(3 Marks)**

**Exercise 4.** It is known that a point charge $Q$ at position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ induces the potential

$$V_x(p) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|x - p|}$$

at $p \in \mathbb{R}^3 \setminus \{x\}$. Suppose three equal point charges are located at the points $x = (0, 0, 0)$, $y = (0, 1, 0)$ and $z = (1, 0, 0)$ in $\mathbb{R}^3$. They induce the potential

$$V(p) = V_x(p) + V_y(p) + V_z(p)$$

at $p \in \mathbb{R}^3 \setminus \{x, y, z\}$. Find all local extrema of $V(p)$ in the $p_1$-$p_2$-plane. You may use a computer algebra program to solve the equations numerically; give the coordinates of the extrema to two decimal place.
**(4 Marks)**

**Exercise 5.** Find the maximum of $f(x, y, z) = z$ under the constraints $x + y + z = 1$ and $z^2/(xy^3) = 3$ (this problem occurs in seeking to maximize the yield of an ammonia reactor).
**(3 Marks)**

**Exercise 6.** A cylindrical storage tank of height $h$ and radius $r$ is to be designed to hold a volume $V$ of water. Find the height and radius that minimise the surface area of the tank.
**(3 Marks)**

**Exercise 7.** Find all extrema (if they exist) of the following real functions $f$ subject to the constraints given.

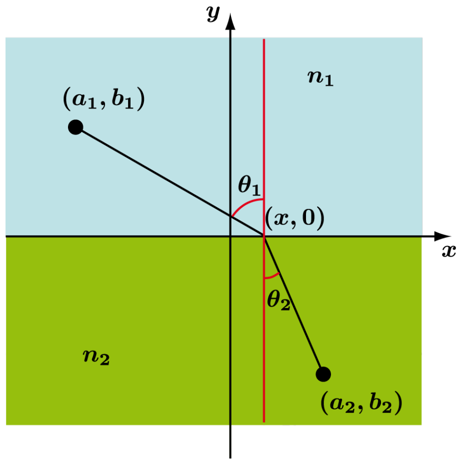
i) $f(x,y) = 4x^2 + 3y^2 - 5xy - 8x$ subject to $x + y = 4$,

ii) $f(x,y) = 4x^2 + 9y^2 + 6y - 4x + 13$ subject to $x - 3y + 3 = 0$,

iii) $f(x,y,z) = (x-1)^2 + (y+2)^2 + (z-2)^2$ subject to $2x + 3y - 1 = 0$, $x + y + 2z - 4 = 0$

**(2 + 2 + 2 Marks)**

**Exercise 8.** In geometrical optics, we consider the problem of refraction at the boundary of two media. Medium 1 has the index of refraction $n_1$, Medium 2 has the index of refraction $n_2$. (The index of refraction of a medium is the quotient of the speed of light in vacuum divided by the speed of light in the medium.)



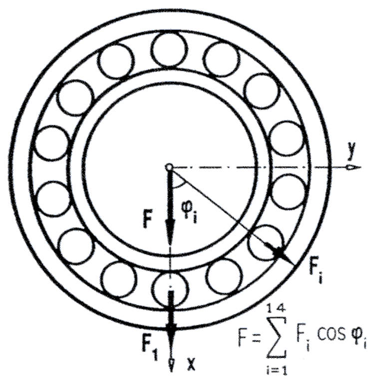

A light ray originating at a point $(a_1, b_1)$ in Medium 1 will travel in a straight line until it meets the interface of the two media, whereupon it will enter Medium 2 and again travel in a straight line to the point $(a_2, b_2)$. By Fermat's principle, the lines are such that the travel time of the light ray will be minimized.

Write the travel time as a function $f$ of $(x, y)$, where $(x, y)$ are the coordinates of the point on the interface of the media where refraction occurs. Note that $f$ needs to be minimized subject to the constraint $g(x,y) = y = 0$. Use the method of Lagrange multipliers to verify Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

**(3 Marks)**

**Exercise 9.** A ball bearing with 14 balls is to be prestressed in such a way that when the interior ring is subjected to a load of $F = 3000\,\mathrm{N}$ the maximal ball deformation is minimized:




Every ball in the bearing is subjected to a radial load, pressed and elastically deformed. This prestress presses the interior ring concentrically against the exterior ring by a distance $e$ (measured in $\mu$m). The load $F$ translates the interior ring downwards by $x$ [$\mu$m].

The radial deformation $\delta_i$ of the $i$th ball is given by

$$\delta_i = e + x\cos\varphi_i, \qquad i = 1, 2, \ldots, 14,$$

in first-order approximation. For $y \in \mathbb{R}$ we define

$$(y)_+ = \begin{cases} y, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

According to the Hertz model, the relationship between the pressing load $F_i$ and the deformation $\delta_i$ is given by

$$F_i = C \cdot (\delta_i)_+^{3/2}$$

where $C$ is a constant depending on the geometry of the bearing. Here, $C = 10\,\mathrm{N}/(\mu\mathrm{m}^{3/2})$. If we set $\varphi_i = 2\pi(i-1)/14$, $i = 1, \ldots, 14$, as in the sketch, we have

$$F(e,x) = \sum_{i=1}^{14} F_i \cos\varphi_i = C \cdot \sum_{i=1}^{14} \left(e + x\cos\frac{\pi(i-1)}{7}\right)_+^{3/2} \cos\frac{\pi(i-1)}{7}$$

The maximal load is borne by ball 1 ($\varphi_1 = 0$),

$$F_1(e,x) = C(e+x)^{3/2}.$$

i) *No prestress:* Set $e = 0$ and calculate $x$ for $F = 3000\,\mathrm{N}$. How large is the maximal ball load $F_1(0, x)$ and how many balls are under load?

ii) *Optimal prestress:* Minimize $F_1(e, x)$ for $e \geq 0$, $x > 0$, under the constraint $F(e, x) = 3000\,\mathrm{N}$. How many balls are under load?

*Hint:* Eliminate the multiplier and find an equation for $\frac{e}{x}$; solve this quation numerically.

The solution will yield that the optimal prestress gives $e = 8.182\,\mu\mathrm{m}$ and $x = 10.88\,\mu\mathrm{m}$. The maximal ball load is reduced by 11.2% under optimal prestress, but there will always be 11 balls under load.
**(2 + 4 Marks)**