UNIVERSITY OF MICHIGAN

UM-SJTU JOINT INSTITUTE
上海交通大学交大密西根
联合学院

SHANGHAI JIAO TONG UNIVERSITY

Applied Calculus III

Exercise Set 12

Date Due: 12:00 PM, Tuesday, the 19th of August 2010

Office hours: Tuesdays, 12:00-1:00 PM and on the SAKAI system

Exercise 1. Calculate the line integrals $\int_S L$ (here $S \subset \mathbb{R}^n$ is a curve that, if no orientation is given, is oriented such that the resulting integral is positive and L is a differential form on $\mathbb{R}^n$):

i) $L = (x^2 - 2xy)\,dx + (y^2 - 2xy)\,dy$ along the parabola $y = x^2$ joining $(-2, 4)$ and $(1, 1)$.

ii) $L = x\,dx + y\,dy + (xz - y)\,dz$ along the straight line from $(0, 0, 0)$ to $(1, 2, 4)$.

iii) $L = x^2y^2\,dx + xy^2\,dy$ along the closed curve consisting of the straight line $x = 1$ and the parabola $y^2 = x$ (oriented positively, i.e., in the counter-clockwise direction).

iv) $L = -y/(x^2 + y^2)\,dx + x/(x^2 + y^2)\,dy$ along the positively oriented circle $x^2 + y^2 = r^2$. Verify that the components L_1 and L_2 of the vector field $L = L_1\,dx + L_2\,dy$ satisfy $\partial_y L_1 = \partial_x L_2$. Why does the integral along the circle not vanish?

$(2 + 2 + 2 + (2 + 1)$ Marks)

Exercise 2. Let $F(x, y) = (2x\cos y, -x^2\sin y)$ be a vector field in $\mathbb{R}^2$. Show that F is a gradient field by checking $\partial_y F_1 = \partial_x F_2$ and find a potential function u for F. Next, calculate the integral of F along the curve given by $\gamma(t) = \left(e^t - 1, t\sin\frac{\pi}{t}\right)$, $t \in [0, 1]$.
$((1 + 2) + 1$ Marks)

Exercise 3. Calculate the integrals

$$\int_G (xy + yz + zx)\,dx\,dy\,dz, \qquad G = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x^2 + y^2 + z^2 \leq 1\}$$

$$\int_{\partial Q} \langle (x, y^2, z^3), N\rangle d\sigma, \qquad Q = [-1, 1]^3, \quad N = \text{outward-pointing normal}$$

i) directly and

ii) using the divergence theorem.

$(2 \times (2 + 2) = 8$ Marks)

Exercise 4. Calculate the circulation $\int_S \langle \operatorname{rot} F, N\rangle\,d\sigma$ of the vector field $F(x, y, z) = (x, x+y, x+y+z)$ directly and using Stokes's theorem. Here S is the surface given by $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and N is the normal vector pointing in positive z-direction.
$(2 + 2$ Marks)