Applied Calculus III

Exercise Set 3

Date Due: 12:00 PM, Thursday, the 3rd of June 2010

Office hours: Tuesdays, 12:00-1:00 PM and on the SAKAI system

Exercise 1. Consider the three maps $T_i\colon \mathbb{R}^3 \to \mathbb{R}^3$, $i = 1, 2, 3$, given by

$$\begin{aligned} T_1\big((x,y,z)\big) &= (x - 3y, -2x + 4y + z, -2y + z), \\ T_2\big((x,y,z)\big) &= (3x + y - 2z, -2x + 5y + z, x + 6y - z), \\ T_3\big((x,y,z)\big) &= (x + y, y + z, x + z). \end{aligned}$$

For each of the three maps,

i) write T_i as a 3×3 matrix,

ii) determine the kernel $\ker T_i := \{v \in \mathbb{R}^3 \colon T_i(v) = 0\}$,

iii) find an orthonormal basis of $\ker T_i$,

iv) determine the range $\operatorname{ran} T_i := \{w \in \mathbb{R}^3 \colon \text{for some } v \in \mathbb{R}^3,\ T_i(v) = w\}$,

v) find an orthonormal basis of $\operatorname{ran} T_i$.

($3 \times (\frac{1}{2} + 1 + 1\frac{1}{2} + 1 + 1\frac{1}{2})$ Marks)

Exercise 2. Let

$$A = \begin{pmatrix} -3 & 1 & 5 \\ 2 & 7 & 1 \\ 1 & -3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 8 & -3 & 1 & 2 \\ -5 & 1 & 2 & 7 \\ 1 & -3 & 8 & 25 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 2 & 3 \\ -20 & -10 & -15 \\ -1 & 7 & -2 \\ -13 & 1 & -11 \end{pmatrix}.$$

i) Find the rank of A, B and C.

ii) Calculate the products AB, BC and CB.

iii) Calculate A^{-1}.

($3 \times (1 + \frac{1}{2}) + \frac{1}{2}$ Mark)

Exercise 3. The *transpose* of a matrix $A = (a_{ij}) \in \mathrm{Mat}(m \times n, \mathbb{C})$ is defined as $A^T := (a_{ji}) \in \mathrm{Mat}(n \times m, \mathbb{C})$. For example,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \qquad \text{gives} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{32} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{pmatrix}.$$

i) Show that for $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ and $x, y \in \mathbb{C}^n$

$$\langle x, Ay \rangle = \langle A^* x, y \rangle$$

where $A^* = \overline{A^T}$ is the (component-wise) complex conjugate of A^T. We write $A^* := \overline{A^T}$ and call A^* the *adjoint* of A.

ii) Show that for $A \in \mathrm{Mat}(m \times n, \mathbb{C})$, $B \in \mathrm{Mat}(n \times l, \mathbb{C})$,

$$(AB)^T = B^T A^T.$$

iii) A matrix $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ such that $A = A^T$ is called *symmetric*. If $A = A^*$ it is called *self-adjoint*. Give an example of a self-adjoint but not symmetric matrix.

iv) A matrix $A \in \mathrm{Mat}(n \times n; \mathbb{R})$ is called *orthogonal* if $A^T = A^{-1}$. Show that the column vectors of an orthogonal matrix are othonormal with respect to the standard scalar product in $\mathbb{R}^n$. If $A \in \mathrm{Mat}(n \times n; \mathbb{C})$ and $A^* = A^{-1}$ the matrix is called *unitary* and its columns are orthogonal w.r.t. the standard scalar product in $\mathbb{C}^n$.

(1 + 1 + 1 + 2 Marks)

Exercise 4. Find real 2×2 matrices with the following properties:

i) $A = A^T = A^{-1},\ A \neq 1$ ii) $B^T = B^{-1},\ B \neq B^T$ iii) $C^T = C^{-1},\ C \neq C^{-1}$

($3 \times \frac{1}{2}$ Mark)

Exercise 5. For $A = (a_{ij})_{i,j=1}^n \in \mathrm{Mat}(n \times n, \mathbb{C})$ we define the *trace* $\operatorname{tr} A := \sum_{i=1}^n a_{ii}$.

i) Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for $A, B \in \mathrm{Mat}(n \times n, \mathbb{C})$ and give an example to show that, in general, $\operatorname{tr}(AB) \neq \operatorname{tr} A \operatorname{tr} B$.

ii) Verify that $\langle A, B \rangle_{\mathrm{tr}} := \operatorname{tr}(A^* B)$ defines a scalar product on $\mathrm{Mat}(n \times m, \mathbb{C})$, $n, m \in \mathbb{N}$. Give an explicit formula for the induced norm $\|A\|_{\mathrm{tr}} := \sqrt{\langle A, A \rangle_{\mathrm{tr}}}$ in terms of the matrix components a_{ij}.

iii) Show that the *Pauli spin matrices*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis of the complex vector space $\mathrm{Mat}(2 \times 2, \mathbb{C})$ and that they are mutually orthogonal with respect to the scalar product $\langle \cdot\,, \cdot \rangle_{\mathrm{tr}}$.

(2 + 2 + 2 Marks)

Exercise 6. Let $R \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection about the line through $y = \binom{3}{2}$. Find the matrix for R.
(3 Marks)

Exercise 7. Let $D_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the rotation by $\theta \in [0, 2\pi)$ in the counter-clockwise direction in $\mathbb{R}^2$.

i) Find the matrix for D_θ.

ii) Prove that $D_\varphi D_\theta = D_{\theta + \phi}$ for any $\phi, \theta \in [0, 2\pi)$.

(2+1 Marks)