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上海交通大学交大密西根
联合学院

UNIVERSITY OF MICHIGAN

SHANGHAI JIAO TONG UNIVERSITY

Applied Calculus III

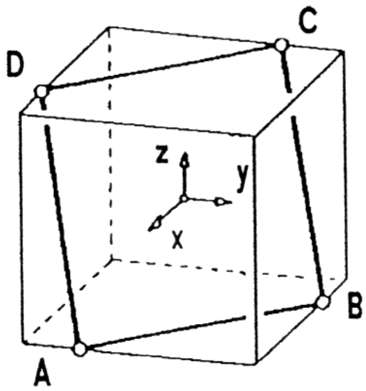
Exercise Set 5

Date Due: 12:00 PM, Tuesday, the 22nd of June 2010

Office hours: Tuesdays, 12:00-1:00 PM and on the SAKAI system

Exercise 1. Consider the cube with the eight corners located at $(\pm1, \pm1, \pm1) \in \mathbb{R}^3$. We select four points on the edges of the cube,

$$A = (1, a, -1), \quad B = (b, 1, -1), \quad C = (-1, -a, 1), \quad D = (-b, -1, 1)$$



i) Show that A, B, C, D are coplanar.

ii) Show that the four points form a parallelogram by verifying that opposite sides are parallel.

iii) Find conditions for the four points to form a square. What is the area of this square?

(1 + 1 + 2 Marks)

Exercise 2. Show that the equation for a circle through three points $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ is given by

$$\det\begin{pmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{pmatrix} = 0.$$

(3 Marks)

Exercise 3. We use the following notation to denote planes P and lines L in $\mathbb{R}^3$:

$$P[u, v; x_0] := \{x \in \mathbb{R}^3 \mid x = x_0 + \alpha \cdot u + \beta \cdot v,\ \alpha, \beta \in \mathbb{R}\}, \qquad L[w; y_0] := \{x \in \mathbb{R}^3 \mid x = y_0 + \alpha \cdot w,\ \alpha \in \mathbb{R}\}$$

for vectors $u, v, w, x_0, y_0 \in \mathbb{R}^3$. (A plane is a two-dimensional, a line a one-dimensional affine subspace of $\mathbb{R}^3$.)

i) Give the equation of the plane containing the three points $\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ and find the line that intersects this plane orthogonally in $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

ii) Find the intersection of $P\left[\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}; \begin{pmatrix}0\\0\\0\end{pmatrix}\right]$ and $P\left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}; \begin{pmatrix}0\\1\\0\end{pmatrix}\right]$.

iii) Find the plane that intersects $L\left[\begin{pmatrix}1\\2\\1\end{pmatrix}; \begin{pmatrix}1\\1\\1\end{pmatrix}\right]$ orthogonally in $\begin{pmatrix}0\\-1\\0\end{pmatrix}$.

(3 + 2 + 2 Marks)

Exercise 4. Show that the equation for a plane through three points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) \in \mathbb{R}^3$ is given by

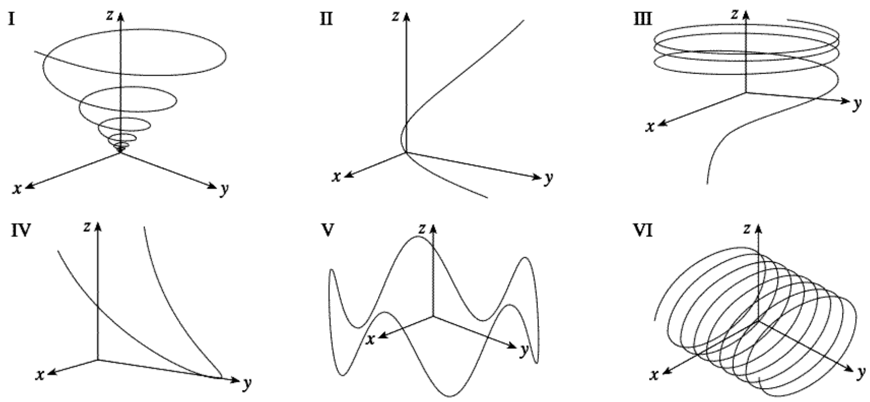
$$\det\begin{pmatrix} x_1-x & y_1-y & z_1-z \\ x_2-x & y_2-y & z_2-z \\ x_3-x & y_3-y & z_3-z \end{pmatrix} = 0.$$

(2 Marks)

Exercise 5. Match the parametrizations $\gamma\colon \mathbb{R} \to \Gamma$ with the curves Γ (labeled I through VI):

$$a)\quad \gamma(t) = \begin{pmatrix} \cos(4t) \\ t \\ \sin(4t) \end{pmatrix}, \qquad b)\quad \gamma(t) = \begin{pmatrix} t \\ t^2 \\ e^{-t} \end{pmatrix}, \qquad c)\quad \gamma(t) = \begin{pmatrix} t \\ 1/(1+t^2) \\ t^2 \end{pmatrix},$$

$$d)\quad \gamma(t) = e^{-t} \begin{pmatrix} \cos(10t) \\ \sin(10t) \\ 1 \end{pmatrix}, \qquad e)\quad \gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ \sin(5t) \end{pmatrix}, \qquad f)\quad \gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ \ln t \end{pmatrix}.$$



(6 × 1/2 Mark)

Exercise 6. For each of the curves $\Gamma \subset \mathbb{R}^n$, $n = 2$ or 3, given by the following parametrizations $\gamma\colon \mathbb{R} \to \Gamma$ do the following:

- Create a sketch (by hand, not computer) and indicate the orientation of the curve by an arrow.
- Determine if the curves are simple, smooth and/or closed.
- Find the unit tangent vector $T \circ \gamma(t)$ at $\gamma(t)$.
- Find the parametric equation for the tangent line at $\gamma(1)$.
- Find the unit normal vector $N \circ \gamma(t)$ at $\gamma(t)$

$$a)\quad \gamma(t) = \begin{pmatrix} t^4+1 \\ t \end{pmatrix}, \qquad b)\quad \gamma(t) = \begin{pmatrix} t^3 \\ t^2 \end{pmatrix}, \qquad c)\quad \gamma(t) = \begin{pmatrix} t \\ \cos(2t) \\ \sin(2t) \end{pmatrix},$$

$$d)\quad \gamma(t) = \begin{pmatrix} 1+t \\ 3t \\ -t \end{pmatrix}, \qquad e)\quad \gamma(t) = \begin{pmatrix} t \\ t \\ \cos t \end{pmatrix}, \qquad f)\quad \gamma(t) = \begin{pmatrix} \sin t \\ \sin t \\ \sqrt{2}\cos t \end{pmatrix}.$$

(6 × (1 + 1/2 + 1 + 1 + 1) Marks)

Exercise 7. The angle between two lines $L[w; y_0]$ and $L[w'; y_0']$ (see Exercise 3) is defined as the angle between w and w'. Suppose that two curves Γ_1 and Γ_2 intersect at some point $x_0 \in \Gamma_1 \cap \Gamma_2$. Then the angle of intersection of the curves at this point is the angle between the tangent vectors at x_0. At which point do the curves parametrized by $\gamma_1, \gamma_2 \colon \mathbb{R} \to \mathbb{R}^3$,

$$\gamma_1(t) = \begin{pmatrix} t \\ 1-t \\ 3+t^2 \end{pmatrix} \qquad \text{and} \qquad \gamma_2(t) = \begin{pmatrix} 3-t \\ t-2 \\ t^2 \end{pmatrix}$$

intersect? Find the angle of intersection at this point.
(2 Marks)

Exercise 8. Let $f, g\colon \mathbb{R} \to \mathbb{R}^n$ be two functions. Then

$$h(x) = \langle f(x), g(x) \rangle$$

defines a function $\mathbb{R} \to \mathbb{R}$. Prove the product rule for the scalar product,

$$h'(x) = \langle f'(x), g(x) \rangle + \langle f(x), g'(x) \rangle.$$

(2 Marks)