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UNIVERSITY OF MICHIGAN

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Applied Calculus III

Exercise Set 7

Date Due: 12:00 PM, Thursday, the 8th of July 2010

Office hours: Tuesdays, 12:00-1:00 PM and on the SAKAI system

Exercise 1. In electrostatics, an *electrostatic potential* $V\colon \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^3$, induces an electric field, whose *electric field intensity* at $p \in \Omega$ is given by

$$E(p) = -\nabla V(p).$$

i) A point charge $Q \in \mathbb{R}$ at the origin induces the electric field

$$V\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \qquad V(x) = \frac{Q}{4\pi\varepsilon_0 |x|},$$

where $\varepsilon_0 > 0$ is the vacuum permittivity. Find the electric field intensity at $p \in \mathbb{R}^3 \setminus \{0\}$.

ii) A dipole $d \in \mathbb{R}^3$ with center at $y \in \mathbb{R}^3$ induces the electric field

$$V\colon \mathbb{R}^3 \setminus \{y\} \to \mathbb{R}, \qquad V(x) = \frac{\langle d, x-y\rangle}{4\pi\varepsilon_0 |x-y|^3}$$

Find the electric field intensity at $p \in \mathbb{R}^3 \setminus \{y\}$.

(1 + 2 Marks)

Exercise 2. In electrostatics, a charge distribution on a curve $\mathcal{C} \subset \mathbb{R}^3$ induces an electric potential in $\mathbb{R}^3 \setminus \mathcal{C}$. The charge distribution is given by the *charge density* $\rho\colon \mathcal{C} \to \mathbb{R}$, and the induced electric potential at $p \in \mathbb{R}^3 \setminus \mathcal{C}$ is given by

$$V(p) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{C}} \frac{\rho}{\operatorname{dist}(p,\,\cdot\,)}$$

where $\operatorname{dist}(p,x) = |p-x|$ denotes the distance between $p \in \mathbb{R}^3 \setminus \mathcal{C}$ and $x \in \mathcal{C}$. An exercise taken from an electromagnetics textbook reads:

> Find the electric field intensity along the axis of a uniform line charge of length L. The uniform line-charge density is ρ_l.

Here a "line charge" is a charge distribution on a straight line, i.e., you may take

$$\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \colon x = y = 0,\ -L/2 \le z \le L/2\}.$$

i) Find the potential and field intensity at points on the z-axis, $|z| > L/2$.

ii) Find the potential and field intensity at points on the x-axis, $|x| > 0$.

Now consider a uniform charge density ρ on the set

$$\mathcal{C} = \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + y^2 = R^2,\ z = 0\}.$$

iii) Find the potential and field intensity at points on the x-axis, $|x| > R$. When x is large ($x \gg R$), compare the field intensity with that induced by a point charge at the origin.

iv) Find the potential and field intensity at points on the z-axis, $|z| > 0$. When z is large ($z \gg R$), compare the field intensity with that induced by a point charge at the origin.

v) Find the potential at the origin.

vi) Find the potential and field intensity at other points in the disk $D = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 < 1,\ z = 0\}$.

(2 + 2 + 3 + 3 + 1 + 3 Marks)

Exercise 3. Check the following functions for differentiability and differentiate them wherever possible.

$$f_1 \colon \mathbb{R} \to \mathbb{R}^3, \quad t \mapsto \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}, \qquad f_2 \colon (0, \infty) \times (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3, \quad (r, \varphi, \vartheta) \mapsto \begin{pmatrix} r\cos\varphi \\ r\sin\varphi\cos\vartheta \\ r\sin\varphi\sin\vartheta \end{pmatrix}$$

$$f_3 \colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto |x|, \qquad f_4 \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \quad x \mapsto \frac{x}{|x|}$$

(4 × 2 Mark)

Exercise 4.

i) Calculate the Jacobian J_f of $f \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$, $f(x, y) = (-y/\sqrt{x^2 + y^2}, x/\sqrt{x^2 + y^2})$.

ii) We define *polar coordinates* through the map $\Phi \colon \mathbb{R}_+ \times (0, 2\pi) \to \mathbb{R}^2 \setminus \{0\}$, $(r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi)$. Calculate the derivative $D\Phi|_{(r,\phi)}$.

iii) Give an explicit expression for the composition $f \circ \Phi$ and calculate $D(f \circ \Phi)|_{(r,\phi)}$.

iv) Verify the chain rule, i.e., $D(f \circ \Phi)|_{(r,\phi)} = Df|_{\Phi(r,\phi)} D\Phi|_{(r,\phi)}$.

(4 × 1 Mark)

Exercise 5. Let $f(x, y)$ be a differentiable function over $[a, b] \times [c, d]$ and let $\alpha, \beta : \mathbb{R} \ni [c, d] \to [a, b] \in \mathbb{R}$ be differentiable. Then the integral

$$I(y) = \int_{\alpha(y)}^{\beta(y)} f(x, y)\, dx$$

is a differentiable function of y in $[c, d]$. Assuming that $\frac{\partial f}{\partial y}(x, y)$ is continuous and $\alpha'(y), \beta'(y)$ exist in $[a, b] \times [c, d]$, give a formula for $I'(y)$.
Hint: Consider the function $I(y_1, y_2, y_3) = \int_{y_1}^{y_2} f(x, y_3)\, dx \colon \mathbb{R}^3 \to \mathbb{R}$ and apply the chain rule.
(3 Points)

Exercise 6. Consider the function $g \colon \mathbb{R}^2 \to \mathbb{R}$,

$$g(x_1, x_2) = \begin{cases} (x_1^2 + x_2^2)\sin((x_1^2 + x_2^2)^{-\frac{1}{2}}), & (x_1, x_2) \neq (0, 0), \\ 0, & (x_1, x_2) = (0, 0). \end{cases}$$

i) Prove directly that g is continuous at the origin $(0, 0)$.

ii) Calculate the partial derivatives $\partial_{x_1} g$ and $\partial_{x_2} g$. Are they continuous at the origin (proof!)?

iii) Is g differentiable at the origin? If yes, find $Dg|_{(0,0)}$. If not, prove that $Dg|_{(0,0)}$ doesnt exist.

(1 + 2 + 2 Marks)

Exercise 7. For any $x \in \mathbb{R}^n \setminus \{0\}$ we have a *polar decomposition* of the form $x = r\omega$, where $r \in (0, \infty)$ and $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ with $|\omega| = 1$. (So ω is on the unit circle $S^{n-1} = \{x \in \mathbb{R}^n \colon |x| = 1\}$). This decomposition is achieved by setting

$$r = |x|, \qquad \omega = \frac{x}{|x|}.$$

Verify that $|\omega| = 1$ and show that for $f \in C(\mathbb{R}^n, \mathbb{R})$,

$$\frac{\partial}{\partial r} f(r\omega) = \sum_{i=1}^{n} \omega_i \left.\frac{\partial f}{\partial x_i}\right|_{x = r\omega}.$$

(3 Marks)