Applied Calculus III

Exercise Set 9

Date Due: 12:00 PM, Thursday, the 22nd of July 2010

Office hours: Tuesdays, 12:00-1:00 PM and on the SAKAI system

Exercise 1. Let $k > 0$ and $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by $f(x_1, x_2) = (x_1^2 + x_2^2)^{-k/2}$.

i) Calculate the integral of f on the ring-shaped domain bounded by two circles of radii $0 < a < b$.

ii) For which values of k does the limit of the integral exist when $a \to 0$?

iii) Give the result of i) and answer ii) when f is replaced with $f\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ given by $f(x_1, x_2, x_3) = (x_1^2+x_2^2+x_3^2)^{-k/2}$ and the ring-shaped domain is replaced by a shell bounded by spheres of radii $0 < a < b$.

(1 + 1 + 2 Marks)

Exercise 2. The shell bounded by spheres of radii $R_a \geq 0$ and $R_b > R_a$ is given by

$$S(R_a, R_b) := \{x \in \mathbb{R}^3 \colon R_a \leq \|x\| \leq R_b\}.$$

Assume that the shell has a constant mass density ϱ and mass equal to M.

i) Calculate the gravitational potential $U(p)$ at a point p with $\|p\| > R_b$.

ii) Calculate the gravitational potential $U(p)$ at a point p with $\|p\| < R_a$.

iii) Let $R_a = 0$, so the shell is a ball of radius R_b. What is the potential at p if $\|p\| < R_a$?

(2 + 2 + 2 Marks)

Exercise 3. In electrostatics, the potential at a point p induced by a charged body B with charge density ϱ is given by

$$V(p) = \frac{1}{4\pi\varepsilon_0} \int_B \frac{\rho}{\operatorname{dist}(p, \cdot\,)}.$$

Let $B = B^2 = \{x \in \mathbb{R}^3 \colon \|x^2\| \leq 1\}$.

i) Calculate the potential $V(r)$ for a point at $(0, 0, r)$, $0 \leq r < \infty$. This will be completely analogous to the corresponding calculations of the gravitational potential in the previous exercise and in the lecture.

ii) The *potential energy of a charged body* B is the electrostatic energy required to "build up" or "put together" the body. It is given by the formula

$$W_e = \frac{1}{2} \int_B \varrho \cdot V,$$

where ϱ is the charge density and V is the potential. Integration is over all points that comprise the body. Use part i) to calculate the potential energy of a uniformly charged sphere of radius R and total charge Q.

iii) An electron has mass $m_e = 9.110 \cdot 10^{-31}$ kg and charge $q_e = 1.602 \cdot 10^{-19}$ C. If the self-energy of the electron is given by $E = m_e c^2$, where $c = 2.998 \cdot 10^8$ m/s is the speed of light in vacuum, and assuming that the electron is a uniformly charged sphere, what would the radius of the electron be? Up to a factor, this radius is called the *classical electron radius*.

(2 + 4 + 2 Marks)

Exercise 4. Let $k > 0$ and $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by $f(x_1, x_2) = (x_1^2 + x_2^2)^{-k/2}$.

i) Calculate the integral of f on the ring-shaped domain bounded by two circles of radii $0 < a < b$.

ii) For which values of k does the limit of the integral exist when $a \to 0$?

iii) Give the result of i) and answer ii) when f is replaced with $f\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ given by $f(x_1, x_2, x_3) = (x_1^2+x_2^2+x_3^2)^{-k/2}$ and the ring-shaped domain is replaced by a shell bounded by spheres of radii $0 < a < b$.

(1 + 1 + 2 Marks)

Exercise 5. Let

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \qquad f(x_1, x_2, x_3) = \sin^2 x_1 \cos x_2 (1 + \sin x_3)$$

Calculate the Taylor polynomial of second order of f at $x = 0$, using

i) the multi-index based formula for functions of multiple variables and

ii) the one-dimensional Taylor expansions of the sine and cosine functions.

(4 + 2 Marks)

Exercise 6. Calculate an approximate value of $1.05^{1.02}$ with an error of less that 10^{-4} by applying Taylor's theorem to the function $f(x, y) = x^y$ at the point $(1, 1)$ with $p = 2$ and estimating the remainder term.
(2 Marks)