Applied Calculus II

Exercise Set 3

Date Due: 10:00 AM, Tuesday, the 19th of October 2010

Office hours: Tuesdays and Thursdays, 12:00-2:00 PM and on the SAKAI system

Exercise 1. Consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}},\ldots.$$

i) Find a recursive representation of the sequence, i.e., a value a_0 and a function f (defined on what domain?) such that $a_{n+1} = f(a_n)$ for all $n \in \mathbb{N}$.

ii) Finden an explicit representation of the sequence and use induction to show that this representation is correct (i.e., it follows from the recursive representation).

iii) Use induction to show that (a_n) is bounded and increasing, so that the limit $a := \lim_{n\to\infty} a_n$ exists. Then calculate the limit.

(1 + 1 + 2 Marks)

Exercise 2.

i) Let (a_n) be a sequence and (a_{2n}), (a_{2n+1}) the subsequences of odd- and even-numbered values of (a_n). Prove that if $a_{2n} \to a$ and $a_{2n+1} \to a$ as $n \to \infty$ then (a_n) converges to a.

ii) Consider the recursively defined subsequence

$$a_1 = 1, \qquad a_{n+1} = 1 + \frac{1}{1 + a_n}.$$

Show that the subsequence (a_{2n}).is monotonic and bounded and therefore converges, i.e., there exists a number $a \in \mathbb{R}$ such that $a_{2n} \to a$. Then prove that $a^2 = 2$ and argue that $a = +\sqrt{2}$.

iii) Do the same for the subsequence (a_{2n+1}). Conclude from i) that $a_n \to \sqrt{2}$. The sequence (a_n) can be written as a *continued fraction*,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}.$$

iv) Prove that for any natural numbers a and b

$$\sqrt{a^2 + b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \cdots}}.$$

(1 + 3 + 2 + 2 Marks)

Exercise 3.

i) Let $a > b \geq 0$ be real numbers and let

$$a_1 = \frac{a+b}{2}, \qquad b_1 = \sqrt{ab}$$

be their *arithmetic* and *geometric mean*, respectively. Show that the recursively defined sequences

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 1, 2, \ldots$$

converge, and that their limits are equal. This limit is called the *arithmetic-geometric mean* of a and b.

You may proceed as follows (but other methods are alsoa cceptable): Show first that (a_n), (b_n) are monotonic (increasing or decreasing?) and hence deduce the existence of $\lim a_n$ and $\lim b_n$. Then estimate $|a_{n+1} - b_{n+1}|$ by $|a_n - b_n|$, and deduce the equality of the limits.

ii) The *harmonic mean* b_1 of $a > b > 0$ is defined by

$$\frac{1}{b_1} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

Let a_1 be the arithmetic mean of a, b, $a > b$ as above and set

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad \frac{1}{b_{n+1}} = \frac{1}{2}\left(\frac{1}{a_n} + \frac{1}{b_n}\right).$$

Show that the sequences $(a_n), (b_n)$ converge to the same limit and calculate this limit

(4 + 3 Marks)

Exercise 4. The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n} \tag{1}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.

i) Show that if (p_n) is convergent, then the only possible values for its limit are 0 and $b - a$.

ii) Show that if $p_0 = 0$, then $p_n = 0$ for all $n \in \mathbb{N}$. Similarly, show that if $b > a$ and $p_0 = b - a$, then $p_n = b - a$ for all $n \in \mathbb{N}$. The points 0 and $b - a$ are *equilibrium points* of (1).

iii) Show that $p_{n+1} < (b/a)p_n$. Prove that if $b < a$ and $p_0 > 0$ then $p_n \to 0$ as $n \to \infty$, i.e., the population dies out.

iv) Now assume that $a < b$. Show that if $p_0 < b - a$ then (p_n) is increasing and $p_n < b - a$ for all $n \in \mathbb{N}$. Show also that if $p_0 > b - a$ then (p_n) is decreasing and $p_n > b - a$ for all $n \in \mathbb{N}$. Deduce that $p_n \to b - a$ whenever $a < b$. The population size $b - a$ is called the *carrying capacity* of the population.

(2 + 2 + 2 + 3 Marks)

Exercise 5. A sequence[1] that arises in ecology as a model for population growth is defined by the *logistic difference equation*

$$p_{n+1} = kp_n(1 - p_n).$$

where p_n measures the size of the population of the nth generation of a single species. To keep the numbers manageable, p_n is a fraction of the maximal size of the population, so $0 \leq p_n \leq 1$. This discrete model is preferable to a continuous model for modeling (e.g.) insect populations, where mating and death occur in a periodic fashion.

An ecologist is interested in predicting the size of the population as time goes on, and asks these questions: Will it stabilize at a limiting value? Will it change in a cyclical fashion? Or will it exhibit random behavior?

Write a program or use computer software (e.g., MatLab, Mathematica, Maple) to compute the first n terms of this sequence starting with an initial population p_0, where $0 < p_0 < 1$. then do the following:

i) Calculate 20 or 30 terms of the sequence for $p_0 = 1/2$ and for two values of k such that $1 < k < 3$. Graph the sequences. Do they appear to converge? Repeat for a different value of p_0 between 0 and 1. Does the limit depend on the choice of p_0? Does it depend on the choice of k?

ii) Calculate terms of the sequence for a value of k between 3 and 3.4 and plot them. What do you notice about the behavior of the terms?

iii) Experiment with values of k between 3.4 and 3.5. What happens to the terms?

iv) For values of k between 3.6 and 4, compute and plot at least 100 terms and comment on the behavior of the sequence. What happens if you change p_0 by 0.001? This type of behavior is called chaotic and is exhibited by insect populations under certain conditions.

(4 + 2 + 2 + 3 Marks)

Exercise 6. Evaluate the following limits, if they exist:

i) $\lim\limits_{x\to 1} \dfrac{x^3 - 1}{x^2 - 1}$, ii) $\lim\limits_{h\to 0} \dfrac{(2+h)^3 - 8}{h}$, iii) $\lim\limits_{h\to 0} \dfrac{\sqrt{1+h} - 1}{h}$,

iv) $\lim\limits_{x\to 2} \dfrac{x^4 - 16}{x - 2}$, v) $\lim\limits_{t\to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2 + t}\right)$, vi) $\lim\limits_{h\to 0} \left(\dfrac{1}{h(3+h)} - \dfrac{1}{3h}\right)$,

vii) $\lim\limits_{x\to 1} \dfrac{\sqrt{x} - x^2}{1 - \sqrt{x}}$, viii) $\lim\limits_{x\to 2} \dfrac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$, ix) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{1+3x} - 1}$

(9 × 1 Marks)

Exercise 7. For fixed $a, b, c \in \mathbb{R}$, find $\alpha, \beta \in \mathbb{R}$, such that

$$\lim_{x\to\infty} \left(\sqrt{ax^2 + bx + c} - \alpha x - \beta\right) = 0.$$

Having found such $\alpha, \beta \in \mathbb{R}$, can there exist different numbers $\alpha', \beta' \in \mathbb{R}$ such that $\lim\limits_{x\to\infty} (\sqrt{ax^2 + bx + c} - \alpha' x - \beta') = 0$? Explain!

(2 + 1 Marks)

Exercise 8. Prove the following statements:

i) $x + x^2 = O(x)$ as $x \to 0$, ii) $\dfrac{1}{x^2 + x} = O\left(\dfrac{1}{x}\right)$ as $x \to 0$, iii) $\dfrac{1}{x} = O\left(\dfrac{1}{x^2}\right)$ as $x \to 0$,

iv) $x + x^2 = O(x^2)$ as $x \to \infty$, v) $\dfrac{1}{x^2 + x} = O\left(\dfrac{1}{x}\right)$ as $x \to \infty$, vi) $\dfrac{1}{x^2} = O\left(\dfrac{1}{x}\right)$ as $x \to \infty$,

vii) $x + x^2 = o(\sqrt{x})$ as $x \to 0$, viii) $\dfrac{1}{x^2 + x} = o\left(\dfrac{1}{x^3}\right)$ as $x \to 0$, ix) $\dfrac{1}{x} = o\left(\dfrac{1}{x^2}\right)$ as $x \to 0$.

(9 × 1 Mark)

[1] see *Stewart*, Lab Project “Logistic Sequences”.