UNIVERSITY OF MICHIGAN

UM-SJTU JOINT INSTITUTE
上海交通大学交大密西根
联合学院

SHANGHAI JIAO TONG UNIVERSITY

Applied Calculus II

Exercise Set 5

Date Due: 4:00 PM, Thursday, the 4th of November 2010

Office hours: Tuesdays and Thursdays, 12:00-2:00 PM and on the SAKAI system

Exercise 1. Find a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\sup_{x\in\mathbb{R}}|f(x)| = 1$ but $\sup_{x\in\mathbb{R}}|f'(x)| = \infty$.
(2 Marks)

Exercise 2. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at $x = 0$ but that f' is not continuous at $x = 0$.
(3 Marks)

Exercise 3. The derivative of a function $f\colon \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$, can itself be regarded as a function $f'\colon \Omega \to \mathbb{R}$. The derivative of f', if it exists, is called the *second derivative* of f and denoted by f''. Calculate the first and second derivatives of the following functions:

$$f(x) = e^{-5x}\cos(3x), \qquad f(x) = \sin(\sin(\sin x)), \qquad f(x) = (1 + \cos^2 x)^6.$$

(3 × 2 Marks)

Exercise 4. The second derivative f'' of a function f is again a function, so it can be differentiated again, to yield the *third derivative* f'''. This procss can be continued, yielding higher derivatives, denoted by $f^{(n)}$, $n = 1, 2, 3, \ldots$.

Use mathematical induction to show the *Leibniz rule* for the nth derivative of the product of two functions f, g that are n times differentiable at $x \in \mathbb{R}$:

$$(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

(3 Marks)

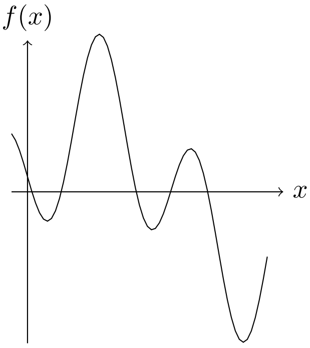
Exercise 5. Calculate the 100th derivative of the real functions $f(x) = (x^2 + 3x + 2)^{-1}$ and $g(x) = \dfrac{x^2+1}{x^3 - x}$.
(2 + 2 Marks)

Exercise 6. Prove that

$$\big(\sin^n x \cos(nx)\big)' = n \sin^{n-1} x \cos(n+1)x, \qquad n \in \mathbb{N} \setminus \{0\}.$$

Then find a similar formula for $\big(\cos^n x \cos(nx)\big)'$.
(3 + 3 Marks)

Exercise 7. The sketch at right shows the graph of a function f. Sketch the graph of f'.
(2 Marks)



Exercise 8. Use the inverse function theorem to find the derivatives of the following functions:

$$f(x) = \arccos x, \qquad f(x) = \arctan x.$$

(2 + 2 Marks)

Exercise 9. If $p(x)$ is the total value of the production when there are x workers in a plant, then the average productivity of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}.$$

i) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?

ii) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.

(2 + 2 Marks)

Exercise 10. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$P'(t) = r_0 \left(1 - \frac{P(t)}{P_c}\right) P(t) - \beta P(t)$$

where $r_0 > 0$ is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and β is the percentage of the population that is harvested.

i) What value of $P'(t)$ corresponds to a stable population?

ii) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.

iii) What happens if β is raised to 5%?

(1/2 + 1 + 1 Marks)

Exercise 11. In the study of ecosystems, predator-prey models are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$C'(t) = aC - bCW, \qquad W'(t) = -cW + dCW,$$

where $ra, b, c, d > 0$.

i) What values of $C'(t)$ and $W'(t)$ correspond to stable populations?

ii) How would the statement "The caribou go extinct" be represented mathematically?

iii) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

(1/2 + 1 + 2 Marks)