University of Michigan

交大密西根学院

UM–SJTU Joint Institute

Shanghai Jiao Tong University

Applied Calculus II

Exercise Set 8

Date Due: 4:00 PM, Thursday, the 25th of November 2010

Office hours: Tuesdays and Thursdays, 12:00-2:00 PM and on the SAKAI system

Exercise 1. Calculate the following integrals:

$$\text{i)} \int_0^1 \ln x\,dx, \qquad \text{ii)} \int \frac{\ln\ln x}{x}\,dx, \qquad \text{iii)} \int e^x \sin x\,dx, \qquad \text{iv)} \int_1^{e^{\pi/2}} \sin\ln x\,dx, \qquad \text{v)} \int_0^{\frac{\pi}{2}} \ln\sin x\,dx,$$

$$\text{vi)} \int \tan x\,dx, \qquad \text{vii)} \int \tan^2 x\,dx, \qquad \text{viii)} \int \frac{1}{x^2(1+x)^2}\,dx, \qquad \text{xi)} \int e^{x^2} x(1+x^2)\,dx, \qquad \text{x)} \int \sqrt{\frac{1-x}{1+x}}\,dx.$$

(10 × 2 Marks)

Exercise 2. Calculate the indefinite integral

$$\int \frac{dx}{ax^2+bx+c}, \qquad a, b, c \in \mathbb{R}.$$

Take care to distinguish two cases according to the sign of $\Delta := 4ac - b^2$.
(4 Marks)

Exercise 3. Let $a_n := \int_0^{\pi/2} \sin^n x\,dx$.

i) Show that $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence.

ii) Prove the recursion formula

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\,dx \qquad (*)$$

and calculate $\int_0^{\pi/2} \sin^2 x\,dx$, $\int_0^{\pi/2} \sin x\,dx$.

iii) Using ii) together with mathematical induction, find expressions for a_{2k} and a_{2k+1}, $k \in \mathbb{N}$. Deduce *Wallis' Product formula*

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}\cdots$$

(1 + 2 + 2 Marks)

Exercise 4. Decide whether or not the following improper integrals exist:

$$\int_0^\infty \frac{1}{\sqrt{1+x^3}}, \qquad \int_0^\infty \frac{x}{1+x^{3/2}}, \qquad \int_0^\infty \frac{1}{x\sqrt{1+x}}.$$

(3 × 1 Mark)

Exercise 5. The *Weierstrass substitution* allows the transformation of every rational integrand containing sine and cosine functions into a purely rational integrand.

i) Show that the substitution $t = \tan\frac{x}{2}$, $-\pi < x < \pi$, gives the following identities

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\,dt. \qquad (*)$$

ii) Use $(*)$, to calculate the integrals a), b) and to prove formula c),

$$\text{a)} \int \frac{1}{\sin x}\,dx \qquad \text{b)} \int \frac{1}{\cos x}\,dx \qquad \text{c)} \int \frac{1}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}} \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right| + \text{const}$$

Note: $\tan\frac{\pi}{4} = 1$, $\tan\frac{\pi}{8} = \sqrt{2} - 1 = (1+\sqrt{2})^{-1}$, $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

(2 + 1 + 2 + 3) Marks)