Applied Calculus II

Exercise Set 9

Date Due: 4:00 PM, Thursday, the 2nd of December 2010

Office hours: Tuesdays and Thursdays, 12:00-2:00 PM and on the SAKAI system

Exercise 1. Through *trigonometric substitution* integrands of the form $\sqrt{\pm x^2 \pm \beta^2}$ may be treated. The substitutin rules are listed below:

Expression	Substitution		Identity
$\sqrt{\beta^2 - x^2}$	$x = \beta \sin\theta,$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{\beta^2 + x^2}$	$x = \beta \tan\theta,$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = 1/\cos^2\theta$
$\sqrt{x^2 - \beta^2}$	$x = \beta/\cos\theta,$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$1/\cos^2\theta - 1 = \tan^2\theta$

Use trigonometric substitution to calculate the indefinite integral

$$I(a) = \int \frac{dx}{\sqrt{a^2 - x^2}}, \qquad\qquad a > 1,$$

where your solution should not include any trigonometric functions but may be expressed in terms of roots and logarithms.
(3 Marks)

Exercise 2. Prove that

$$\int_0^1 \frac{1}{x^\alpha}\,dx$$

converges if $\alpha < 1$ and diverges if $\alpha \ge 1$.
(2 Marks)

Exercise 3. If a function $f\colon [a,b] \to \mathbb{R}$ is unbounded near a point $x_0 \in (a,b)$ the improper integral $\int_a^b f(x)\,dx$ may not exist. In this case one may still attempt to make sense of the integral by defining the *Cauchy principal value* of the integral as

$$\mathrm{p.\,v.} \int_a^b f := \lim_{\varepsilon \searrow 0} \left(\int_a^{x_0 - \varepsilon} f + \int_{x_0+\varepsilon}^b f \right),$$

i) Show that $\mathrm{p.\,v.} \int_{-1}^1 \frac{1}{x}\,dx = 0$.

ii) Show that $\mathrm{p.\,v.} \int_{-1}^1 \frac{f(x)}{x}\,dx = 0$ if f is an even function.

iii) Calculate $\mathrm{p.\,v.} \int_{-1}^1 \frac{f(x)}{x}\,dx$ for $f(x) = x^2 - 2x + 1$.

iv) Will the principal value $\mathrm{p.\,v.} \int_{-1}^1 \frac{f(x)}{x}\,dx$ exist for any differentiable function f? Why or why not? Explain!

(1 + 1 + 2 + 3 Marks)

Exercise 4. Use the Mean Value Theorem (of differential calculus) to prove the *Mean Value Theorem of integral calculus*: Let $[a, b] \subset \mathbb{R}$ be a closed interval and $f\colon [a, b] \to \mathbb{R}$ a real function. Then there exists a $\xi \in (a, b)$ such that

$$\int_a^b f(x)\, dx = (b - a) f(\xi).$$

(2 Marks)

Exercise 5. We consider the so-called *complete elliptic integral of the first kind*,

$$T(a, b) := \frac{2}{\pi} \int\limits_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}, \qquad a, b > 0.$$

(This integral plays an important role when computing the exact period of a pendulum. We will study this at a later date.)

i) Substitute $t := b \tan \theta$ to obtain

$$T(a, b) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}}.$$

ii) Substitute $u := (t - ab/t)/2$ to obtain

$$T(a, b) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{du}{\sqrt{\left(\left(\frac{a+b}{2}\right)^2 + u^2\right)\left(ab + u^2\right)}}.$$

iii) Let $M(a, b)$ denote the arithmetic-geometric mean of $a, b > 0$ (see Exercise 3 on Exercise Set 3). Use i) and ii) above to show that

$$T(a, b) = \frac{1}{M(a, b)}.$$

(2 + 2 + 3 Marks)

Exercise 6. Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine barrels. (In fact, Kepler published a book *Stereometria doliorum* in 1715 devoted to methods for finding the volumes of barrels.) They often approximated the shape of the sides by parabolas.

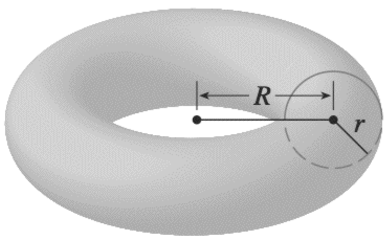
i) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the parabola $y = R - cx^2$, $x \in [-\frac{h}{2}, \frac{h}{2}]$, $c > 0$. Show that the radius of each end of the barrel is $r = R - d$, where $d = ch^2/4$.

ii) Show that the volume enclosed by the barrel is

$$V = \frac{\pi h}{3} \left(2R^2 + r^2 - \frac{2}{5} d^2\right)$$

(2 + 4 Marks)

Exercise 7. By integrating the areas of the circular cross-sections, find the volume of the torus pictured below:



(3 Marks)