Vv556 Methods of Applied Mathematics I

Linear Operators

Assignment 4

Date Due: 2:00 PM, Thursday, the 19th of October 2017

This assignment has a total of **(14 Marks)**.

Exercise 4.1
Calculate the Fourier-sine series of the function f defined on $[0, \pi]$ and given by $f(x) = x(\pi - x)$. Evaluate the series at a suitable point to find the value of the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

(4 Marks)

Exercise 4.2

i) Show that
$$\mathcal{B} = \left\{ \frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}} \cos(nx) \right\}_{n=1}^{\infty}$$
is an orthonormal system in $L^2([0, \pi])$.
(2 Marks)

ii) Show[1] that $\cos^k x$, $k \in \mathbb{N}$, is a linear combination of $\{1, \cos x, \cos(2x), \dots, \cos(kx)\}$, so that
$$\operatorname{span}\{1, \cos x, \cos^2 x, \dots, \cos^k x\} = \operatorname{span}\{1, \cos x, \cos(2x), \dots, \cos(kx)\}.$$
(2 Marks)

iii) Let $f \in C([0, \pi])$. Consider the change of variables $y = \cos x$, $x \in [0, \pi]$, and define $\widetilde{f} \in C([-1, 1])$ by $\widetilde{f}(y) = f(\arccos y)$. Use the Weierstraß Approximation Theorem to approximate uniformly $\widetilde{f}$ by polynomials, and show that this means that f can be approximated uniformly by finite linear combinations of $1, \cos x, \cos(2x), \dots$.
(2 Marks)

iv) Conclude that $\operatorname{span} \mathcal{B}$ is dense in $C([0, \pi])$ in the $\|\cdot\|_\infty$-norm.
(2 Marks)

v) Further deduce that $\operatorname{span} \mathcal{B}$ is dense in $C([0, \pi])$ in the $\|\cdot\|_{L^2}$-norm.
(1 Mark)

vi) Further deduce that $\operatorname{span} \mathcal{B}$ is dense in $L^2([0, \pi])$ in the $\|\cdot\|_{L^2}$-norm (use that $C([0, \pi])$ is by definition dense in $L^2([0, \pi])$). Hence, $\mathcal{B}$ is an orthonormal basis of $L^2([-\pi, \pi])$.
(1 Mark)

[1] See *Stakgold*, Ex. 4.6.3