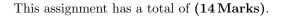
# Vv556 Methods of Applied Mathematics I

**Linear Operators** 

# Assignment 4

Date Due: 2:00 PM, Thursday, the 19<sup>th</sup> of October 2017



## Exercise 4.1

Calculate the Fourier-sine series of the function f defined on  $[0, \pi]$  and given by  $f(x) = x(\pi - x)$ . Evaluate the series at a suitable point to find the value of the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

### (4 Marks)

#### Exercise 4.2

i) Show that

$$\mathcal{B} = \left\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\cos(nx)\right\}_{n=1}^{\infty}$$

is an orthonormal system in  $L^2([0,\pi])$ . (2 Marks)

ii) Show<sup>1</sup> that  $\cos^k x, k \in \mathbb{N}$ , is a linear combination of  $\{1, \cos x, \cos(2x), \dots, \cos(kx)\}$ , so that

 $span\{1, \cos x, \cos^2 x, \dots, \cos^k x\} = span\{1, \cos x, \cos(2x), \dots, \cos(kx)\}.$ 

#### (2 Marks)

- iii) Let  $f \in C([0,\pi])$ . Consider the change of variables  $y = \cos x$ ,  $x \in [0,\pi]$ , and define  $\tilde{f} \in C([-1,1])$  by  $\tilde{f}(y) = f(\arccos y)$ . Use the Weierstraß Approximation Theorem to approximate uniformly  $\tilde{f}$  by polynomials, and show that this means that f can be approximated uniformly by finite linear combinations of  $1, \cos x, \cos(2x), \ldots$  (2 Marks)
- iv) Conclude that span  $\mathcal{B}$  is dense in  $C([0, \pi])$  in the  $\|\cdot\|_{\infty}$ -norm. (2 Marks)
- v) Further deduce that span  $\mathcal{B}$  is dense in  $C([0, \pi])$  in the  $\|\cdot\|_{L^2}$ -norm. (1 Mark)
- vi) Further deduce that span  $\mathcal{B}$  is dense in  $L^2([0,\pi])$  in the  $\|\cdot\|_{L^2}$ -norm (use that  $C([0,\pi])$  is by definition dense in  $L^2([0,\pi])$ ). Hence,  $\mathcal{B}$  is an orthonormal basis of  $L^2([-\pi,\pi])$ . (1 Mark)



<sup>&</sup>lt;sup>1</sup>See Stakgold, Ex. 4.6.3