Vv556 Methods of Applied Mathematics I

Linear Operators

Assignment 6

Date Due: 2:00 PM, Thursday, the 2nd of November 2017

This assignment has a total of **(10 Marks)**.

Exercise 6.1

A *Hilbert-Schmidt operator* on $L^2([a,b])$ has the form

$$(Ku)(x) := \int_a^b k(x,y)u(y)\,dy,$$

where the kernel k satisfies $\int_a^b \int_a^b |k(x,y)|^2\,dx\,dy =: M^2 < \infty$.

Now consider the operator L on $L^2([0,1])$ defined by

$$(Lf)(x) = \int_0^x f(y)\,dy.$$

i) Show that L is a Hilbert-Schmidt operator and that $\|L\| \le 1/\sqrt{2}$ from a basic Hilbert-Schmidt estimate. (you just need to follow Examples 6 and 7 on pages 300/301 of Stakgold's book).
(2 Marks)

ii) Recall that $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$, where

$$e_n = \sqrt{2}\cos\left(\frac{2n+1}{2}\pi x\right),$$

is an orthonormal basis on $L^2([0,1])$. Use the basis expansion of f in terms of $\mathcal{B}$ to show that $\|Lf\| \le (2/\pi)\|f\|$.
(2 Marks)

iii) Show that $\|L\| = 2/\pi$.
(3 Marks)

iv) Calculate the matrix elements of L with respect to $\mathcal{B}$.
(2 Marks)

v) Find the adjoint of L.
(1 Mark)