Vv556 Methods of Applied Mathematics I

Linear Operators

Assignment 7

Date Due: 2:00 PM, Thursday, the 9th of November 2017

This assignment has a total of **(18 Marks)**.

Exercise 7.1

Consider again the left- and right-shift operators L and R on ℓ^2. The goal of this exercise is to show directly that the numbers $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ lie in the continuous spectrum of these operators.

An eigenvector for the left-shift operator satisfying $Le_\lambda = \lambda \cdot e_\lambda$ is given by

$$e_\lambda = (1, \lambda, \lambda^2, \lambda^3, \dots).$$

However, for $|\lambda| = 1$ this vector is not in ℓ^2. Consider instead the "almost-eigenvector"

$$e_\lambda^{(N)} := \frac{1}{N+1}(1, \lambda, \lambda^2, \dots, \lambda^N, 0, \dots)$$

for $|\lambda| = 1$.

i) Show that $\|e_\lambda^{(N)}\|_2 = 1$.
(0.5 Marks)

ii) Calculate $(L - \lambda I)e_\lambda^{(N)}$ and show that $\|(L - \lambda I)e_\lambda^{(N)}\|_2 \to 0.$ as $N \to \infty$.
(1 Mark)

iii) Deduce that $\lambda \in \sigma_{\text{continuous}}(L)$.
(0.5 Marks)

iv) Find a sequence of almost-eigenvectors for the right-shift operator R and show that all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ lie in $\sigma_{\text{continuous}}(R)$.
(2 Marks)

Exercise 7.2

Let $L\colon L^2([0,1]) \to L^2([0,1])$ be given by

$$(Lu)(x) = x \cdot u(x)$$

i) Show that the domain of the inverse $(L^{-1}u)(x) = \frac{1}{x}u(x)$ is dense. *Hint:* the domain surely includes the set

$$M = \Big\{ u \in L^2([0,1])\colon \underset{\varepsilon>0}{\exists} \underset{0\le x\le\varepsilon}{\forall} u(x) = 0 \Big\}.$$

and you can show that M is dense by hand.
(2 Marks)

ii) Show that L is bounded, find $\|L\|$ and verify that L^{-1} is unbounded.
(3 Marks)

iii) Find the state of L and of L^{-1}.
(2 Marks)

iv) Is L self-adjoint?
(1 Mark)

v) Find the spectrum of L.
(3 Marks)

Exercise 7.3
On $L^2([0,1])$ consider the operator L defined by

$$(Lf)(x) = \int_0^x f(y)\,dy.$$

i) Show that the state of L is $(\text{II}, 1_n)$. *Hint:* does the range of L contain the set of polynomials?
(2 Marks)

ii) Find the adjoint of L.
(1 Mark)

The following exercise is just to fill a gap in the lecture slides. There is no need to complete it.
Exercise 7.4
For two sequences $(a_n), (b_n) \in \ell^1$ the *Cauchy product* of their series is defined as

$$\Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big) = \sum_{n=0}^{\infty} c_n$$

where

$$c_n := \sum_{i+j=n} a_i b_j = \sum_{i=0}^{n} a_i b_{n-i}.$$

(It can be shown that the equality holds and that $(c_n) \in \ell^1$.) The sequence (c_n) is said to be the *convolution* of (a_n) and (b_n) and we write

$$(c_n) = (a_n) * (b_n).$$

Prove *Young's convolution inequality*:

$$\|(a_n) * (b_n)\|_p \leq \|(a_n)\|_p \cdot \|(b_n)\|_1 \qquad 1 \leq p < \infty.$$

Instructions: for the case $p > 1$, write

$$|c_n| \leq \sum_{i=0}^{n} \big(|a_i| \cdot |b_{n-i}|^{1/p}\big) \cdot |b_{n-i}|^{1/q}$$

and apply Hölder's inequality.