

Vv556 Methods of Applied Mathematics I

Linear Operators

Assignment 8

Date Due: 2:00 PM, Thursday, the 16th of November 2017



This assignment has a total of (20 Marks).

Exercise 8.1

Define the operator $L: \ell^2 \rightarrow \ell^2$ by

$$L(x_n) = \left(x_1, \frac{1}{\sqrt{2}}x_2, \frac{1}{\sqrt{3}}x_3, \frac{1}{\sqrt{4}}x_4, \dots \right).$$

- i) Show that L is not a Hilbert-Schmidt operator.
(1 Mark)
- ii) Show that L is self-adjoint.
(1 Mark)
- iii) Show that L is compact.
(2 Marks)
- iv) Find upper and lower bounds for the spectrum of L .
(2 Marks)
- v) Find the spectrum of L .
(4 Marks)

Exercise 8.2

Let $M := \{u \in L^2([0, 1]): u \in C^2(0, 1), u(0) = u(1) = 0\}$ and

$$L = \frac{d^2}{dx^2}.$$

Let $K: L^2([0, 1]) \rightarrow L^2([0, 1])$ be given by

$$(Ku)(x) := \int_0^1 g(x, \xi)u(\xi) d\xi$$

with

$$g(x, \xi) := \begin{cases} x(1 - \xi) & x < \xi, \\ \xi(1 - x) & x \geq \xi. \end{cases}$$

- i) Show that K is the inverse of L , i.e., $KL = I$ on M . (This requires some elementary calculations with the integral.)
(2 Marks)
- ii) Show that L is unbounded.
(2 Marks)
- iii) Show that K is compact.
(2 Marks)
- iv) Show that $g(x, \xi) = g(\xi, x)$ and deduce that K is self-adjoint.
(2 Marks)
- v) Find the upper and lower bounds of the Rayleigh quotient for K .
(2 Marks)