

# Vv556 Methods of Applied Mathematics I

## Linear Operators

### Assignment 9

Date Due: 2:00 PM, Thursday, the 23<sup>rd</sup> of November 2017



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This assignment has a total of (14 Marks).

#### Exercise 9.1

Let  $M := \{u \in L^2([0, 1]): u \in C^2(0, 1), u(0) = u(1) = 0\}$  and define

$$L = \frac{d^2}{dx^2}$$

on  $M \subset L^2([0, 1])$ . Let  $K: L^2([0, 1]) \rightarrow L^2([0, 1])$  be given by

$$(Ku)(x) := \int_0^1 g(x, \xi)u(\xi) d\xi$$

with

$$g(x, \xi) := \begin{cases} x(1 - \xi) & x < \xi, \\ \xi(1 - x) & x \geq \xi. \end{cases}$$

It is known that  $K$  is compact and self-adjoint. Furthermore,  $K = L^{-1}$ .

- i) Show that  $L = K^{-1}$  if  $K$  is restricted to  $M$ . (You will have to perform some careful differentiations; use the chain rule.)  
(2 Marks)
- ii) Consider the eigenvalue equation  $Lu = \lambda u$  for  $u \in M$ . Show that  $\lambda \neq 0$  is an eigenvalue for  $L$  if and only if  $1/\lambda$  is an eigenvalue for  $K$ . How are the eigenfunctions of  $K$  and  $L$  related?  
(2 Marks)
- iii) Find the eigenvalues  $\lambda_n \in \mathbb{C}$  and eigenfunctions  $\psi_n \in M$  of  $L$  by solving an ordinary differential equation. You will have to prove first that  $\lambda_n \in \mathbb{R}$  and then consider the cases  $\lambda_n > 0$ ,  $\lambda_n = 0$  and  $\lambda_n < 0$ .  
(5 Marks)
- iv) Find the spectrum of  $K$ . Is zero an eigenvalue? Find the sequence of eigenvalues  $(\lambda_n)$  such that  $\lambda_n \searrow 0$ .  
(3 Marks)
- v) According to the spectral theorem,  $L^2([0, 1])$  has an orthonormal basis consisting of eigenfunctions of  $K$ . Use this to deduce that a certain Fourier basis is actually a basis.  
(2 Marks)