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Entanglement dynamics of two-qubit pure state

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Abstract We show that the entanglement dynamics for the pure state of a closed two-qubit system is part of a 10-dimensional complex linear differential equation defined on a supersphere, and the coefficients therein are completely determined by the system Hamiltonian. We apply the result to two physical examples of Josephson junction qubits and exchange Hamiltonians, deriving analytic solutions for the time evolution of entanglement. The Hamiltonian coefficients determine whether the entanglement is periodic. These results allow of investigating how to generate and manipulate entanglements efficiently, which are required by both quantum computation and quantum communication.

Keywords Quantum entanglement · Entanglement dynamics · Two-qubit system · Dynamical equation

1 Introduction

Entanglement is one of the most striking quantum mechanical properties that plays a central role in quantum computation and quantum communication [1]. It is a major resource used in many applications such as quantum algorithm, teleportation, and

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quantum cryptography. In recent years, there are several problems receiving considerable research interests, including dynamical evolution of entanglement where the objective is to find out how the entanglement of a quantum system evolves as time elapses [2,3]. This is especially important to quantum information processing that relies primarily upon the generation, manipulation, and detection of quantum entanglement.

Many of the current researches on this topic focus on the entanglement decay or production for an open quantum system interacting with the surrounding environment. For example, Yu and Eberly [4–6] revealed that quantum entanglement influenced by environmental noise can undergo a sudden death. In Ref. [7], Konrad et al. proved a factorization law for bipartite system that describes entanglement evolutions with a noisy channel. Ref. [8] studied the time evolution of entanglement of qubits interacting with a common environment. Cui et al. [9] discussed the time evolution of entanglement in bipartite systems and revealed that the entanglement sudden death can appear in both open and closed systems and is dependent on the initial condition. See Refs. [4–18] and the references therein for full details.

Here we consider a different problem: for a given closed two-qubit quantum system with Hamiltonian H, what are dynamics characterized by differential equations that governs the time evolution of its pure state entanglement? To our knowledge, this is a basic yet to be answered question. In many applications of quantum computation and quantum communication, often desired is to efficiently generate entanglement from some initial state. Therefore, it is of particular value to investigate how the entanglement of a qubit system evolves as a function of time so as to analyze the capability of a quantum system to produce and further manipulate quantum entanglement. Moreover, it will also help us to have a deeper understanding of the fundamentals of various entanglement phenomena such as entanglement sudden death and rebirth [6].

We will show that for a closed two-qubit system, its pure state entanglement dynamics can be described by part of a 10-dimensional complex differential equation. All the coefficients in this equation are determined by the Hamiltonian. For specific two-qubit systems of Josephson junction and exchange Hamiltonians, we will derive closed-form solutions for the time evolution of entanglement.

2 Backgrounds

First we briefly introduce some mathematical backgrounds (See Ref. [19] for details). The quantum operations for a two-qubit system are defined on the special unitary Lie group $SU(4)$. The associated Lie algebra is denoted as $\mathfrak{su}(4)$ and has a direct sum decomposition $\mathfrak{su}(4) = \mathfrak{p} \oplus \mathfrak{k}$, where

$$\begin{aligned} \mathfrak{k} &= \text{span}\, \frac{i}{2}\left\{\sigma_x^1,\ \sigma_y^1,\ \sigma_z^1,\ \sigma_x^2,\ \sigma_y^2,\ \sigma_z^2\right\}, \\ \mathfrak{p} &= \text{span}\, \frac{i}{2}\left\{\sigma_x^1\sigma_x^2,\ \sigma_x^1\sigma_y^2,\ \sigma_x^1\sigma_z^2,\ \sigma_y^1\sigma_x^2,\ \sigma_y^1\sigma_y^2,\right. \\ &\qquad \left.\sigma_y^1\sigma_z^2,\ \sigma_z^1\sigma_x^2,\ \sigma_z^1\sigma_y^2,\ \sigma_z^1\sigma_z^2\right\}. \end{aligned} \tag{1}$$

Here σ_x, σ_y, and σ_z are the Pauli matrices, and $\sigma_\alpha^1\sigma_\beta^2 = \sigma_\alpha \otimes \sigma_\beta$. The set $\mathfrak{k}$ contains the local terms, whereas $\mathfrak{p}$ has the nonlocal or coupling terms. An arbitrary Hamiltonian for two-qubit system can be represented by a linear combination of the basis matrices in Eq. (1) as

$$\begin{aligned} H = &\frac{a_1}{2}\sigma_x^1 + \frac{a_2}{2}\sigma_y^1 + \frac{a_3}{2}\sigma_z^1 + \frac{a_4}{2}\sigma_x^2 + \frac{a_5}{2}\sigma_y^2 + \frac{a_6}{2}\sigma_z^2 \\ &+\frac{a_7}{2}\sigma_x^1\sigma_x^2 + \frac{a_8}{2}\sigma_x^1\sigma_y^2 + \frac{a_9}{2}\sigma_x^1\sigma_z^2 + \frac{a_{10}}{2}\sigma_y^1\sigma_x^2 + \frac{a_{11}}{2}\sigma_y^1\sigma_y^2 \\ &+ \frac{a_{12}}{2}\sigma_y^1\sigma_z^2 + \frac{a_{13}}{2}\sigma_z^1\sigma_x^2 + \frac{a_{14}}{2}\sigma_z^1\sigma_y^2 + \frac{a_{15}}{2}\sigma_z^1\sigma_z^2, \end{aligned} \tag{2}$$

where a_k's are all real numbers. Denote the state of the quantum system as ψ. The dynamics of ψ is determined by the Schrödinger equation:

$$\dot{\psi} = iH\psi \tag{3}$$

with initial state $\psi(t_0)$, where t_0 is the initial time.

We use the concurrence of ψ as an entanglement measure, which is defined in Refs. [20,21] as

$$C(\psi) = |\,\mathrm{Ent}\,\psi|,$$

where $\mathrm{Ent}\,\psi = \psi^T\sigma_y^1\sigma_y^2\,\psi$. It can be shown that the concurrence $C(\psi)$ is invariant under the local operations and it ranges from 0 to 1. The condition $C(\psi) = 0$ holds true if and only if ψ is an unentangled state. In the case when $C(\psi)$ achieves maximal value 1, such ψ is called a maximally entangled state. The sufficient and necessary condition for ψ to be maximally entangled were analyzed in Ref. [19].

3 Entanglement dynamics

The concurrence $C(\psi)$ defines a measure of entanglement for the two-qubit pure state ψ. In what follows, we will derive the dynamics of $\mathrm{Ent}\,\psi$, that is, the differential equation that governs its time evolution. To this end, take derivative of $\mathrm{Ent}\,\psi$:

$$\begin{aligned} \frac{d}{dt}\,\mathrm{Ent}\,\psi &= \dot{\psi}^T\sigma_y^1\sigma_y^2\psi + \psi^T\sigma_y^1\sigma_y^2\dot{\psi} \\ &= i\psi^T\left(H^T\sigma_y^1\sigma_y^2 + \sigma_y^1\sigma_y^2 H\right)\psi \\ &= i\psi^T\big(a_{11}I - a_{15}\sigma_x^1\sigma_x^2 - a_7\sigma_z^1\sigma_z^2 - a_8 i\sigma_z^1 + a_{12}i\sigma_x^2 \\ &\quad + a_{13}\sigma_x^1\sigma_z^2 + a_{14}i\sigma_x^1 + a_9\sigma_z^1\sigma_x^2 - a_{10}i\sigma_z^2\big)\psi. \end{aligned} \tag{4}$$

The derivative of $\mathrm{Ent}\,\psi$ depends on the terms such as $\psi^T\psi$ and $\psi^T\sigma_x^1\sigma_x^2\psi$ in Eq. (4). To get the complete dynamics of $\mathrm{Ent}\,\psi$, we need to find the derivatives of all these terms. For the ease of notation, define

$$x_k = \psi^T P_k \psi, \quad k = 1, \ldots, 10, \tag{5}$$

where

$$\begin{aligned} &P_1 = \sigma_y^1 \sigma_y^2, \quad P_2 = I, \qquad P_3 = \sigma_x^1 \sigma_x^2, \\ &P_4 = \sigma_z^1 \sigma_z^2, \quad P_5 = i\sigma_z^1, \quad P_6 = i\sigma_x^2, \\ &P_7 = \sigma_x^1 \sigma_z^2, \quad P_8 = i\sigma_x^1, \quad P_9 = \sigma_z^1 \sigma_x^2, \quad P_{10} = i\sigma_z^2. \end{aligned} \tag{6}$$

It is clear that $x_1 = \text{Ent}\,\psi$, and Eq. (4) can be rewritten as

$$\begin{aligned} \dot{x}_1 = i(&a_{11}x_2 - a_{15}x_3 - a_7x_4 - a_8x_5 + a_{12}x_6 \\ &+ a_{13}x_7 + a_{14}x_8 + a_9x_9 - a_{10}x_{10}). \end{aligned}$$

For the other x_k's, we can similarly obtain

$$\dot{x}_k = \dot{\psi}^T P_k \psi + \psi^T P_k \dot{\psi} = i\psi^T (H^T P_k + P_k H)\psi. \tag{7}$$

One salient feature of the matrices $\{P_j\}_{j=1}^{10}$ defined in Eq. (6) is that any matrix in the form of $H^T P_k + P_k H$ can be represented by a linear combination of all these matrices. However, they do not form a subalgebra of $\mathfrak{su}(4)$. Now, $\dot{x}_k$ in Eq. (7) can be written as a linear combination of all the x_j's.

Let $x = [x_1, \ldots, x_{10}]^T$. After some derivations, we obtain that x satisfies the following linear differential equation

$$\dot{x} = iAx, \tag{8}$$

where

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{12}^\dagger & A_{22} & A_{23} \\ \hline A_{13}^\dagger & A_{23}^\dagger & A_{33} \end{array}\right] \tag{9}$$

$$= \left[\begin{array}{cccc|ccc|ccc} 0 & a_{11} & -a_{15} & -a_7 & -a_8 & a_{12} & a_{13} & a_{14} & a_9 & -a_{10} \\ a_{11} & 0 & a_7 & a_{15} & -ia_3 & -ia_4 & a_9 & -ia_1 & a_{13} & -ia_6 \\ -a_{15} & a_7 & 0 & -a_{11} & a_{10} & -ia_1 & ia_5 & -ia_4 & ia_2 & a_8 \\ -a_7 & a_{15} & -a_{11} & 0 & -ia_6 & -a_{14} & -ia_2 & -a_{12} & -ia_5 & -ia_3 \\ \hline -a_8 & ia_3 & a_{10} & ia_6 & 0 & a_{13} & a_{12} & -ia_2 & ia_4 & a_{15} \\ a_{12} & ia_4 & ia_1 & -a_{14} & a_{13} & 0 & -a_8 & a_7 & ia_3 & ia_5 \\ a_{13} & a_9 & -ia_5 & ia_2 & a_{12} & -a_8 & 0 & -ia_6 & a_{11} & -ia_1 \\ \hline a_{14} & ia_1 & ia_4 & -a_{12} & ia_2 & a_7 & ia_6 & 0 & -a_{10} & a_9 \\ a_9 & a_{13} & -ia_2 & ia_5 & -ia_4 & -ia_3 & a_{11} & -a_{10} & 0 & a_{14} \\ -a_{10} & ia_6 & a_8 & ia_3 & a_{15} & -ia_5 & ia_1 & a_9 & a_{14} & 0 \end{array}\right]. \tag{10}$$

The block matrices A_{ij}'s in Eq. (9) have a conforming partition as those in Eq. (10). The elements in the k-th row of A are exactly those coefficients to represent $H^T P_k + P_k H$

as a linear combination of $\{P_j\}_{j=1}^{10}$. Equations (8) and (9) reveal that the dynamics for pure state entanglement of a closed two-qubit system is part of a 10-dimensional complex linear differential equations. We can also split the real and imaginary parts of x to get a real differential equation with dimension 20. This is the key result of this paper.

Note that the matrix A is Hermitian, i.e., $A = A^\dagger$. Then,

$$\frac{d}{dt}x^\dagger x = \dot{x}^\dagger x + x^\dagger \dot{x} = -ix^\dagger A^\dagger x + x^\dagger i A x = 0,$$

which yields that

$$\|x(t)\|^2 = x^\dagger(t)x(t) = x^\dagger(t_0)x(t_0),$$

for all $t \geq t_0$. The norm of $x(t)$ is thus conserved along the trajectory of the Schrödinger equation (3). We now show that this conserved quantity is 2. Let the initial state of ψ be

$$\psi(t_0) = [\psi_1 + i\psi_2, \psi_3 + i\psi_4, \psi_5 + i\psi_6, \psi_7 + i\psi_8]^T, \tag{11}$$

where ψ_l's are real numbers and satisfy $\sum_{l=1}^{8} \psi_l^2 = 1$. Let the initial condition of x_k be

$$x_k(t_0) = p_k + iq_k, \tag{12}$$

i.e., p_k and q_k are the real and imaginary parts of $x_k(t_0)$, respectively. Substituting Eq. (11) into Eq. (5), we can represent p_k and q_k in terms of ψ_l's as

$$\begin{aligned}
p_1 &= 2(\psi_3\psi_5 - \psi_4\psi_6 - \psi_1\psi_7 + \psi_2\psi_8),\\
p_2 &= \psi_1^2 - \psi_2^2 + \psi_3^2 - \psi_4^2 + \psi_5^2 - \psi_6^2 + \psi_7^2 - \psi_8^2,\\
p_3 &= 2(\psi_3\psi_5 - \psi_4\psi_6 + \psi_1\psi_7 - \psi_2\psi_8),\\
p_4 &= \psi_1^2 - \psi_2^2 - \psi_3^2 + \psi_4^2 - \psi_5^2 + \psi_6^2 + \psi_7^2 - \psi_8^2,\\
p_5 &= 2(-\psi_1\psi_2 - \psi_3\psi_4 + \psi_5\psi_6 + \psi_7\psi_8),\\
p_6 &= 2(-\psi_2\psi_3 - \psi_1\psi_4 - \psi_6\psi_7 - \psi_5\psi_8),\\
p_7 &= 2(\psi_1\psi_5 - \psi_2\psi_6 - \psi_3\psi_7 + \psi_4\psi_8),\\
p_8 &= 2(-\psi_2\psi_5 - \psi_1\psi_6 - \psi_4\psi_7 - \psi_3\psi_8),\\
p_9 &= 2(\psi_1\psi_3 - \psi_2\psi_4 - \psi_5\psi_7 + \psi_6\psi_8),\\
p_{10} &= 2(-\psi_1\psi_2 + \psi_3\psi_4 - \psi_5\psi_6 + \psi_7\psi_8),
\end{aligned} \tag{13}$$

and

$$
\begin{aligned}
q_1 &= 2(\psi_4\psi_5 + \psi_3\psi_6 - \psi_2\psi_7 - \psi_1\psi_8),\\
q_2 &= 2(\psi_1\psi_2 + \psi_3\psi_4 + \psi_5\psi_6 + \psi_7\psi_8),\\
q_3 &= 2(\psi_4\psi_5 + \psi_3\psi_6 + \psi_2\psi_7 + \psi_1\psi_8),\\
q_4 &= 2(\psi_1\psi_2 - \psi_3\psi_4 - \psi_5\psi_6 + \psi_7\psi_8),\\
q_5 &= \psi_1^2 - \psi_2^2 + \psi_3^2 - \psi_4^2 - \psi_5^2 + \psi_6^2 - \psi_7^2 + \psi_8^2,\\
q_6 &= 2(\psi_1\psi_3 - \psi_2\psi_4 + \psi_5\psi_7 - \psi_6\psi_8),\\
q_7 &= 2(\psi_2\psi_5 + \psi_1\psi_6 - \psi_4\psi_7 - \psi_3\psi_8),\\
q_8 &= 2(\psi_1\psi_5 - \psi_2\psi_6 + \psi_3\psi_7 - \psi_4\psi_8),\\
q_9 &= 2(\psi_2\psi_3 + \psi_1\psi_4 - \psi_6\psi_7 - \psi_5\psi_8),\\
q_{10} &= \psi_1^2 - \psi_2^2 - \psi_3^2 + \psi_4^2 + \psi_5^2 - \psi_6^2 - \psi_7^2 + \psi_8^2.
\end{aligned}
\tag{14}
$$

Proceeding further, we have

$$
\|x(t)\| = \sqrt{x^\dagger(t_0)x(t_0)} = \sqrt{\sum_{k=1}^{10}\left(p_k^2 + q_k^2\right)} = 2\sum_{l=1}^{8}\psi_l^2 = 2.
$$

Therefore, the dynamics of Eq. (8) is defined on a supersphere with radius 2. At this point, the physical interpretation of this quantity remains unclear to us.

For a general Hamiltonian in Eq. (2), there exists a local operation $k \in SU(2) \otimes SU(2)$ such that all the coupling terms in $kHk^\dagger$ vanish except $\sigma_x^1\sigma_x^2$, $\sigma_y^1\sigma_y^2$, and $\sigma_z^1\sigma_z^2$ [19]:

$$
\begin{aligned}
kHk^\dagger = {} & \frac{a_1}{2}\sigma_x^1 + \frac{a_2}{2}\sigma_y^1 + \frac{a_3}{2}\sigma_z^1 + \frac{a_4}{2}\sigma_x^2 + \frac{a_5}{2}\sigma_y^2 + \frac{a_6}{2}\sigma_z^2\\
& + \frac{a_7}{2}\sigma_x^1\sigma_x^2 + \frac{a_{11}}{2}\sigma_y^1\sigma_y^2 + \frac{a_{15}}{2}\sigma_z^1\sigma_z^2.
\end{aligned}
\tag{15}
$$

By an abuse of notation, we again use a_k to denote the coefficients in the right hand side of Eq. (15). Because

$$
e^{iHt}\psi(t_0) = k^\dagger(e^{ikHk^\dagger})k\psi(t_0)
$$

and the local operation $k^\dagger$ does not change the entanglement of quantum state, the function Ent ψ generated by H with initial state $\psi(t_0)$ is the same as that by $kHk^\dagger$ with $k\psi(t_0)$. We then only need to study a simplified differential equation, where all the entries in A corresponding to cross-coupling terms are 0. In particular, two diagonal blocks A_{22} and A_{33} in Eq. (9) both become zero matrices.

4 Physical examples

To illustrate the ideas, we now study the dynamic equation of pure state entanglement for two physical examples, namely, charge-coupled Josephson junction and exchange Hamiltonians. We focus on the Hamiltonians rather than physical implementation details, because the Hamiltonian completely determines the entanglement dynamics.

First consider a charge-coupled Josephson junction qubit system discussed in Ref. [22]. The Hamiltonian is given by

$$H_1 = -\frac{E_J}{2}\left(\sigma_x^1 + \sigma_x^2\right) + \frac{E_J^2}{E_L}\sigma_y^1\sigma_y^2, \tag{16}$$

which contains both local and nonlocal terms. Then $a_1 = a_4 = -E_J$ and $a_{11} = 2E_J^2/E_L$. Setting all the other a_k's to 0 in Eq. (9), we obtain a reduced order differential equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_6 \\ \dot{x}_8 \end{bmatrix} = i \left[\begin{array}{ccc|ccc} 0 & a_{11} & 0 & 0 & 0 & 0 \\ a_{11} & 0 & 0 & 0 & -ia_1 & -ia_1 \\ 0 & 0 & 0 & -a_{11} & -ia_1 & -ia_1 \\ \hline 0 & 0 & -a_{11} & 0 & 0 & 0 \\ 0 & ia_1 & ia_1 & 0 & 0 & 0 \\ 0 & ia_1 & ia_1 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_6 \\ x_8 \end{bmatrix}.$$

Let $\alpha = E_J/E_L$. Then $a_{11} = -2\alpha a_1$. Solving the differential equation above yields

$$\text{Ent}\,\psi(t) = x_1(t) = r_1(t) + is_1(t),$$

where

$$\begin{aligned} r_1(t) = {} & \frac{(p_1 - p_4) + (q_6 + q_8)\alpha}{2(1+\alpha^2)} + \frac{(q_2 + q_3)\alpha}{2\sqrt{1+\alpha^2}} \sin\sqrt{1+\alpha^2}E_J t \\ & + \frac{(p_1 - p_4)\alpha - (q_6 + q_8)}{2(1+\alpha^2)}\alpha\cos\sqrt{1+\alpha^2}E_J t \\ & + \frac{(p_1 + p_4)}{2}\cos\alpha E_J t + \frac{(q_2 - q_3)}{2}\sin\alpha E_J t, \end{aligned} \tag{17}$$

and

$$\begin{aligned} s_1(t) = {} & \frac{(q_1 - q_4) - (p_6 + p_8)\alpha}{2(1+\alpha^2)} - \frac{(p_2 + p_3)\alpha}{2\sqrt{1+\alpha^2}} \sin\sqrt{1+\alpha^2}E_J t \\ & + \frac{(q_1 - q_4)\alpha + (p_6 + p_8)}{2(1+\alpha^2)}\alpha\cos\sqrt{1+\alpha^2}E_J t \\ & + \frac{(q_1 + q_4)}{2}\cos\alpha E_J t - \frac{(p_2 - p_3)}{2}\sin\alpha E_J t. \end{aligned} \tag{18}$$

Here p_k's and q_k's are defined in Eqs. (13) and (14). Hence,

$$C(\psi) = \sqrt{r_1^2(t) + s_1^2(t)}.$$

The entanglement evolution has two frequency components $\sqrt{1+\alpha^2}E_J$ and αE_J. When the ratio between these two values, $\sqrt{1+\alpha^2}/\alpha$, is a rational number, entanglement is periodic; otherwise, it is aperiodic.

Next let us consider two-qubit exchange Hamiltonian $H_2 = \frac{1}{2}(a_7\sigma_x^1\sigma_x^2 + a_{11}\sigma_y^1\sigma_y^2 + a_{15}\sigma_z^1\sigma_z^2)$. In this case, all the local terms vanish, i.e., $a_1 = 0$, ..., $a_6 = 0$. For the block matrices in Eq. (9), we have $A_{12} = A_{13} = 0$. Therefore, the dynamics of $x_u = [x_1, x_2, x_3, x_4]^T$ is decoupled from that of $[x_5, \ldots, x_{10}]^T$, which leads to

$$\dot{x}_u = iA_{11}x_u, \tag{19}$$

or more explicitly,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = i \begin{bmatrix} 0 & a_{11} & -a_{15} & -a_7 \\ a_{11} & 0 & a_7 & a_{15} \\ -a_{15} & a_7 & 0 & -a_{11} \\ -a_7 & a_{15} & -a_{11} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{20}$$

Because A_{11} is also a Hermitian (or symmetric indeed) matrix, the norm of x_u is also conserved. Let

$$T = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

The matrix A_{11} can be diagonalized as

$$\begin{aligned} T^{-1}A_{11}T = \mathrm{diag}\{&a_7 - a_{11} + a_{15},\ -a_7 - a_{11} - a_{15}, \\ &-a_7 + a_{11} + a_{15},\ a_7 + a_{11} - a_{15}\}. \end{aligned} \tag{21}$$

The entanglement evolution therefore has four frequency components as given in Eq. (21). We can obtain the same diagonal matrix if transforming the Hamiltonian H_2 into the Bell basis. We then have

$$\mathrm{Ent}\,\psi(t) = x_1(t) = \left(r^T(t) + is^T(t)\right) l,$$

where $l = [p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4]^T$, and

$$\begin{aligned} r^T(t) = [\,&\cos a_7 t \cos a_{11} t \cos a_{15} t, \quad \sin a_7 t \cos a_{11} t \sin a_{15} t, \\ &-\sin a_7 t \sin a_{11} t \cos a_{15} t, \, -\cos a_7 t \sin a_{11} t \sin a_{15} t, \\ &\sin a_7 t \sin a_{11} t \sin a_{15} t, \, -\cos a_7 t \sin a_{11} t \cos a_{15} t, \\ &\cos a_7 t \cos a_{11} t \sin a_{15} t, \quad \sin a_7 t \cos a_{11} t \cos a_{15} t], \end{aligned}$$

$$\begin{aligned} s^T(t) = [\,&\sin a_7 t \sin a_{11} t \sin a_{15} t, \quad \cos a_7 t \sin a_{11} t \cos a_{15} t, \\ &-\cos a_7 t \cos a_{11} t \sin a_{15} t, -\sin a_7 t \cos a_{11} t \cos a_{15} t, \\ &\cos a_7 t \cos a_{11} t \cos a_{15} t, \quad \sin a_7 t \cos a_{11} t \sin a_{15} t, \\ &-\sin a_7 t \sin a_{11} t \cos a_{15} t, \, -\cos a_7 t \sin a_{11} t \sin a_{15} t]. \end{aligned}$$

Therefore,

$$C(\psi) = \sqrt{l^T \left(r(t) r^T(t) + s(t) s^T(t) \right) l}.$$

Examining the frequency components in Eq. (21), we know that if the pairwise ratios between a_7, a_{11}, and a_{15} are all rational numbers, the entanglement measure is a periodic function.

For specific exchange Hamiltonian, we can further simplify the entanglement measure. For instance, for two-dimensional XY exchange Hamiltonian $H_3 = \sigma_x^1 \sigma_x^2 + \sigma_y^1 \sigma_y^2$, we have

$$\begin{aligned} C^2(\psi) = &\frac{1}{4} \left((p_2 - p_4) \sin 2t + (q_1 + q_3) \cos 2t + q_1 - q_3 \right)^2 \\ &+ \frac{1}{4} \left((q_2 - q_4) \sin 2t - (p_1 + p_3) \cos 2t - p_1 + p_3 \right)^2. \end{aligned}$$

For Ising Hamiltonian $H_4 = \sigma_x^1 \sigma_x^2$, we have

$$C(\psi) = \sqrt{(p_4 \sin t - q_1 \cos t)^2 + (q_4 \sin t + p_1 \cos t)^2}.$$

It is evident that in these two cases, the concurrence measure $C(\psi)$ are both periodic functions.

5 Conclusions

In summary, we have derived the dynamical equation that governs the time evolution of pure state entanglement for closed two-qubit systems. This turns out to be a 10-dimensional differential equation defined on a supersphere. We applied the result to investigate two physical applications, namely, Josephson junction and exchange Hamiltonians. For both cases, we derived analytic solutions for the concurrence measure of entanglement. The coefficients in the Hamiltonian completely determine whether or not the entanglement is a periodic function. We expect to extend the result to mixed state and open systems in the future.

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