Discrete Mathematics

Assignment 1

Date Due: 8:00 PM, Thursday, the 26th of May 2011

Office hours: Tuesdays, 1:00-3:00 PM, and Wednesdays, 12:00-1:00 PM

You are required to compose your solutions in *neat and legible handwriting.* Up to 10% of the total score may be deducted solely due to the apearance and legibility of your writing and your use of the English language.

In order to obtain the highest possible score, make sure that you explain your reasoning. Often, simple formulae are not enough to answer a question. *Explain what you are doing!* This will also ensure that you get a large fraction of the total points even if you make a mistake in your calculations. In short write simple, whole grammatical sentences that include a subject, verb and object.

Exercise 1.

i) Let a, b be statements. Write out the truth tables to prove *de Morgan's rules*:

$$\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b, \qquad\qquad \neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b.$$

ii) Let M be a set and $A, B, C \subset M$. Prove the following equalities:

$$(A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}}, \qquad\qquad (A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}}.$$

(2 + 2 Marks)

Exercise 2. Suppose that a truth table in n propositional variables is specified. Show that a compound proposition with this truth table can be formed by taking the disjunction of conjunctions of the variables or their negations, with one conjunction for each combination of values for which the compound proposition is true. The resulting compound proposition is said to be in *disjunctive normal form.*
(2 Marks)

Exercise 3. A collection of logical operators is called *functionally complete* if every compound proposition is logiacally equivalent to a compound proposition involving only these logical operators.

i) Show that $\{\wedge, \vee, \neg\}$ is a functionally complete collection of logical operators. (Hint: use the disjunctive normal form.)

ii) Show that $\{\wedge, \neg\}$ is a functionally complete collection of logical operators. (Hint: use a de Morgan law.)

iii) Show that $\{\vee, \neg\}$ is a functionally complete collection of logical operators.

(1 + 1 + 1 Marks)

Exercise 4. Let M be a set and $X, Y, Z \subset M$. We define the *symmetric difference*

$$X \bigtriangleup Y := (X \cup Y) \setminus (X \cap Y).$$

i) Prove that $X \bigtriangleup Y = (X \setminus Y) \cup (Y \setminus X)$.

ii) Prove that $X^{\mathrm{c}} \bigtriangleup Y^{\mathrm{c}} = X \bigtriangleup Y$.

iii) Show that the symmetric difference is associative, i.e., $(X \bigtriangleup Y) \bigtriangleup Z = X \bigtriangleup (Y \bigtriangleup Z)$.

iv) Prove that $X \cap (Y \bigtriangleup Z) = (X \cap Y) \bigtriangleup (X \cap Z)$.

(1 + 1 + 2 + 3 Marks)

Exercise 5. The logical operation associated to the symmetric difference is called the *exclusive or*, written as $\oplus$ in logic or XOR in logic gate design. It is defined by the truth table

A	B	$A \oplus B$
T	T	F
T	F	T
F	T	T
F	F	F

i) If $X = \{x\colon A(x)\}$ and $Y = \{x\colon B(x)\}$ show that

$$x \in X \triangle Y \leftrightarrow A(x) \oplus B(x).$$

ii) Express $\oplus$ by logical conjunction, disjunction and negation, i.e., through the operations $\{\wedge, \vee, \neg\}$.

iii) As discussed in the exercises below, any binary operation can be represented through $\{\wedge, \vee, \neg\}$. For technical reasons, it is preferable to represent logical operations using $\{\wedge, \oplus, \neg\}$ instead. Write $\vee$, $\Rightarrow$ and $\Leftrightarrow$ using $\{\wedge, \oplus, \neg\}$.

(2 + 2 + 2 Marks)

Exercise 6. In computer design, the logical operations NAND and NOR play an important role. In logic, NAND is represented by the *Scheffer stroke* $|$ while NOR is represented by the *Peirce arrow* $\downarrow$. They are defined as

$$A \mid B :\equiv \neg(A \wedge B), \qquad\qquad A \downarrow B :\equiv \neg(A \vee B).$$

i) Give the truth tables for $A \mid B$ and $A \downarrow B$.

ii) Prove that $A \downarrow A \equiv \neg A$ and $(A \downarrow B) \downarrow (A \downarrow B) \equiv A \vee B$. Deduce that $\{\downarrow\}$ is a functionally complete collection of logical operators.

iii) Represent the exclusive or $\oplus$ solely through $\downarrow$.

iv) Prove that $\{|\}$ is a functionally complete collection of logical operators.

v) Show that $|$ is not associative, i.e., $(A \mid B) \mid C \not\equiv A \mid (B \mid C)$.

(1 + 2 + 1 + 2 + 1 Marks)

Exercise 7. Sometimes the number of times that an element occurs in an unordered collection matters. *Multisets* are unordered collections of elements where an element can occur as a member more than once. The notation $\{m_1 \cdot a_1, m_2 \cdot a_2, \dots, m_r \cdot a_r\}$ denotes the multiset with element a_1 occurring m_1 times, element a_2 occurring m_2 times, and so on. The numbers m_i, $i = 1, 2, \dots, r$ are called the multiplicities of the elements a_i, $i = 1, 2, \dots, r$.
Let P and Q be multisets. The union of the multisets P and Q is the multi set where the multiplicity of an element is the maximum of its multiplicities in P and Q. The intersection of P and Q is the multiset where the multiplicity of an element is the minimum of its multiplicities in P and Q. The difference of P and Q is the multiset where the multiplicity of an element is the multiplicity of the element in P less its multiplicity in Q unless this difference is negative, in which case the multiplicity is O. The sum of P and Q is the multiset where the multiplicity of an element is the sum of multiplicities in P and Q. The union, intersection, and difference of P and Q are denoted by $P \cup Q$, $P \cap Q$, and $P \setminus Q$, respectively (where these operations should not be confused with the analogous operations for sets). The sum of P and Q is denoted by $P + Q$.
Suppose that A is the multiset that has as its elements the types of computer equipment needed by one department of a university where the multiplicities are the number of pieces of each type needed, and B is the analogous multi set for a second department of the university. For instance, A could be the multiset $\{107 \cdot \text{personal computers}, 44 \cdot \text{routers}, 6 \cdot \text{servers}\}$ and B could be the multiset $\{14 \cdot \text{personal computers}, 7 \cdot \text{routers}, 3 \cdot \text{mainframes}\}$.

i) What combination of A and B represents the equip- ment the university should buy assuming both departments use the same equipment?

ii) What combination of A and B represents the equip- ment that will be used by both departments if both departments use the same equipment?

iii) What combination of A and B represents the equip- ment that the second department uses, but the first department does not, if both departments use the same equipment?

iv) What combination of A and B represents the equip- ment that the university should purchase if the depart- ments do not share equipment?

(4 × 1 Mark)

Exercise 8. Let M be a set and $O_1, O_2 \subset M$. Let $T = (O_1 \cap O_2) \cup (O_1^c \cap O_2^c)$. Show that $(O_1^c \cap O_2) \subset T^c$.
(2 Marks)

Exercise 9. *Fuzzy logic* is used in artificial intelligence. In fuzzy logic, a proposition has a truth value that is a number between 0 and 1, inclusive. A proposition with a truth value of 0 is false and one with a truth value of 1 is true. Truth values between 0 and 1 indicate varying degrees of truth. For example, we might decide assign the following truth values:

- A: "The real function f defined on the interval $[-1, 1]$ given by $f(x) = |x| - 1/3$ is positive" might have a truth value of $2/3$,
- B: "The real function f defined for all real numbers given by $f(x) = \sin x$ is positive" might have a truth value of $1/2$.

In fuzzy logic,

- the truth value of the negation of a statement in fuzzy logic is 1 minus the truth value of the statement.
- the truth value of the conjunction of two statements is the minimum of the truth values of the two statements.
- the truth value of the disjunction of two statements is the maximum of the truth values of the two statements.

Given A and B as above, find the truth values of

i) $A \mid B$,

ii) $A \Rightarrow B$,

iii) $A \oplus B$.

(1 + 1 + 1 Marks)

Exercise 10. *Fuzzy sets* are defined as sets $S = \{x \colon P(x)\}$ where for any x the statement $P(x)$ is assigned a truth value as in the previous exercise. This truth value is then *degree of membership* of x in S. If we list the elements of S, we do so together with their degrees of membership. Consider, for example,

$$A = \{0.1\,a,\ 0.4\,b,\ 0.2\,c\}, \qquad\qquad B = \{0.4\,a,\ 0.7\,c,\ 0.9\,d\}$$

Thus the statement $a \in A$ has a truth value of 0.1. The complement of a fuzzy set S is a fuzzy set S^c where the degree of membership in S^c is 1 minus the degree of membership in S. Similarly, given two fuzzy sets S and T, the truth value of the statement $x \in S \cap T$ is the truth value of $(x \in S) \wedge (x \in T)$. The truth value of $x \in S \cup T$ is the truth value of $(x \in S) \vee (x \in T)$. Given A and B as above and assuming $A, B \subset \{a, b, c, d, e\}$, find

i) B^c,

ii) $(A \cap B)^c$,

iii) $A \triangle B$.

(1/2 + 1/2 + 1 Marks)