Discrete Mathematics

Assignment 10

Date Due: 8:00 PM, Thursday, the 28th of July 2011

Office hours: Tuesdays, 12:00-2:00 PM, and Wednesdays, 12:00-1:00 PM

Exercise 1. Use induction in $(\mathbb{Z}_+^2, \preccurlyeq)$, the set of pairs of natural numbers with lexicographic ordering induced by the ordering $\leq$ of $\mathbb{Z}_+$, to show that if the numbers $a_{m,n}$, $m, n \in \mathbb{Z}_+$, are defined recursively by

$$a_{m,n} = \begin{cases} 5 & m = n = 1, \\ a_{m-1,n} + 2 & n = 1 \wedge m > 1, \\ a_{m,n-1} + 2 & n > 1, \end{cases}$$

then $a_{m,n} = 2(m+n) + 1$ for all $m, n \in \mathbb{Z}_+$.
(3 Marks)

Exercise 2. Let $(S, \preccurlyeq)$ be a poset. We say an element $y \in S$ *covers* and element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs (x, y) such that y covers x is called the *covering relation* of $(S, \preccurlyeq)$.

i) Find the covering relation for $(S, |)$ with $S = \{1, 2, 3, 4, 6, 12\}$.

ii) Find the covering relation for $(\mathcal{P}(S), \subset)$ where $\mathcal{P}(S)$ is the powerset of $S = \{1, 2, 3, 4\}$.

iii) Show that (x, y) belongs to the covering relation of $(S, \preccurlyeq)$ if and only if x is lower than y and there is an edge joining x and y in the Hasse diagram of this poset.

iv) Show that a finite poset can be reconstructed from its covering relation by showing that it is the reflexive transitive closure of the covering relation.

(2 + 2 + 2 + 3 Marks)

Exercise 3. Consider the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

i) Draw the Hasse diagram for this poset.

ii) Find all maximal and minimal elements.

iii) Find the least and greatest elements of the poset, if they exist.

iv) Find all upper bounds of $\{2, 9\}$ and $\sup\{2, 9\}$, if it exists.

v) Find all lower bounds of $\{60, 72\}$ and $\inf\{60, 72\}$, if it exists.

(5 × 1 Marks)

Exercise 4. Which of the following posets are latices?

i) $(\{1, 5, 25, 125\}, |)$

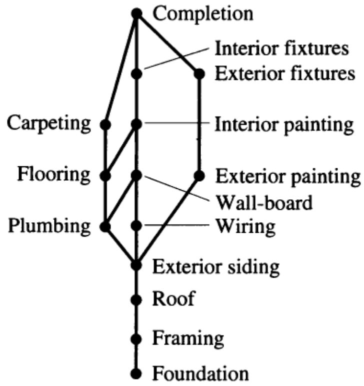
ii) $(\{1, 3, 6, 9, 12\}, |)$

iii) $(\mathbb{Z}, \geq)$

iv) $(\mathcal{P}(S), \subset)$, where $\mathcal{P}(S)$ is the power set of a set S.

(4 × 1 Marks)

Exercise 5. Schedule the tasks needed to build a house, by specifying their order, if the Hasse diagram representing those tasks is as shown in the figure at right.
(3 Marks)



Exercise 6. Let $(S, \preccurlyeq)$ be a poset. We say that $(S, \preccurlyeq)$ is *well-founded* if there does not exist a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in S$ such that $x_{n+1} \prec x_n$ for $n \in \mathbb{N}$. We say that $(S, \preccurlyeq)$ is *dense* if for all $x, y \in S$ with $x \prec y$ there exists a $z \in S$ such that $x \prec z \prec y$.

i) Show that $(\mathbb{Z}, \preccurlyeq)$ with $x \prec y \Leftrightarrow |x| < |y|$ is a well-pounded poset. Sketch (part of) the Hasse diagram for $(\mathbb{Z}, \preccurlyeq)$. Show that $(\mathbb{Z}, \preccurlyeq)$ is not well-ordered.

ii) Show that a dense poset with at least two elements that are comparable is not well-founded.

iii) Show that $(\mathbb{Q}, \leq)$ is a dense poset.

iv) Show that the set of all bit strings with lexicographic order is neither well-founded nor dense.

v) Show that a poset is well-ordered if and only if it is well-founded and totally ordered.

(5 × 2 Marks)

Exercise 7. Let $(S, \preccurlyeq)$ be a well-founded poset. The *principle of well-founded induction* states that $P(x)$ is true for all $x \in S$ if

$$\underset{x\in S}{\forall} \left(\underset{y\in S}{\forall} \big(y \prec x \Rightarrow P(y)\big) \Rightarrow P(x) \right) \tag{$*$}$$

i) Show that no induction basis is needed, i.e., $P(u)$ is true for all minimal eleents of S if $(*)$ holds.

ii) Prove that the principle of well-founded inducton is valid.

iii) Use the principle of well-founded induction on the well-founded poset $(\mathbb{Z}, \preccurlyeq)$, $x \prec y \Leftrightarrow |x| < |y|$, with the product rule of differentiation to show that

$$\frac{d}{dx}x^n = \begin{cases} 0 & \text{for } n = 0, \\ nx^{n-1} & \text{for } n \in \mathbb{Z} \setminus \{0\} \end{cases}$$

(1 + 3 + 2 Marks)