University of Michigan

## Discrete Mathematics

## Assignment 11

Date Due: 8:00 PM, Thursday, the $4^{\text {th }}$ of August 2011
Office hours: Tuesdays, 12:00-2:00 PM, and Wednesdays, 12:00-1:00 PM

Exercise 1. In the following graphs, find the number of vertices, the number of edges and the degree of each vertex. Identify all isolated and pendant vertices. Classify each graph as a simple graph, a multigraph or a pseudograph.
i)

ii)

iii)

(3×2 Marks)
Exercise 2. In the following graphs, determine which ones are bipartite and give a bipartition for those that are.
i)

ii)

iii)

(3×1 Mark)
Exercise 3. Find the union of the graphs ii) and iii) in Exercise 2. Assume that edges between identical vertices are identical.
(2 Marks)
Exercise 4. Give the adjacency matrices for all three (pseudo-)graphs in Exercise 1.
( $3 \times 1$ Marks)
Exercise 5. Draw undirected graphs for the given adjacency matrices:
i) $\left(\begin{array}{lll}1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0\end{array}\right)$
ii) $\left(\begin{array}{lllll}0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3\end{array}\right)$
( $2 \times 2$ Marks)

Exercise 6. Determine whether each given pair of graphs is isomorphic. Exhibit an isomorphism (and prove that it actually is an isomorphism) or prove that no isomorphism exists.
i)


ii)


iii)


iv)


( $4 \times 2$ Marks)
Exercise 7. For which values of $n$ do the following graphs have an Euler circuit?
i) $K_{n}$
ii) $W_{n}$
iii) $C_{n}$
iv) $Q_{n}$

## ( $4 \times 1$ Marks)

Exercise 8. Let $P_{1}$ and $P_{2}$ be two simple paths between the vertices $u$ and $v$ in the simple graph $G$ that do not contain the same set of edges. Show that there is a simple circuit in $G$.
(3 Marks)
Exercise 9. The parts of this exercise outline a proof of Ore's Theorem. Suppose that $G$ is a simple graph with $n$ vertices, $n>3$, and $\operatorname{deg}(x)+\operatorname{deg}(y)>n$ whenever $x$ and $y$ are nonadjacent vertices in $G$. Ore's Theorem states that under these conditions, $G$ has a Hamilton circuit.
i) Show that if $G$ does not have a Hamilton circuit, then there exists another graph $H$ with the same vertices as $G$, which can be constructed by adding edges to $G$ such that the addition of a single edge would produce a Hamilton circuit in $H$.
Hint: Add as many edges as possible at each successive vertex of $G$ without producing a Hamilton circuit.
ii) Show that there is a Hamilton path in $H$.
iii) Let $v_{1}, v_{2}, \ldots, v_{n}$ be a Hamilton path in $H$. Show that $\operatorname{deg}\left(v_{1}\right)+\operatorname{deg}\left(v_{n}\right)>n$ and that there are at most $\operatorname{deg}\left(v_{1}\right)$ vertices not adjacent to $v_{n}$ (including $v_{n}$ itself).
iv) Let $S$ be the set of vertices preceding each vertex adjacent to $v_{1}$ in the Hamilton path. Show that $S$ contains $\operatorname{deg}\left(v_{1}\right)$ vertices and $v_{n} \notin S$.
v) Show that $S$ contains a vertex $v_{k}$, which is adjacent to $v_{n}$, implying that there are edges connecting $v_{1}$ and $v_{k+1}$ and $v_{k}$ and $v_{n}$.
vi) Show that part (iii) implies that $v_{1}, v_{2}, \ldots, v_{k-1}, v_{k}, v_{n}, v_{n-1}, \ldots, v_{k+1}, v_{1}$ is a Hamilton circuit in $G$. Conclude from this contradiction that Ore's Theorem holds.

$$
(2+1+2+2+1+1 \text { Marks })
$$

