University of Michigan

交大密西根学院
UM–SJTU Joint Institute

Shanghai Jiao Tong University

Discrete Mathematics

Assignment 2

Date Due: 8:00 PM, Thursday, the 2nd of June 2011

Office hours: Tuesdays, 1:00-3:00 PM, and Wednesdays, 12:00-1:00 PM

Exercise 1. Show that in defining the natural numbers as follows,

$$\operatorname{succ}(n) := \{n, \emptyset\}, \qquad \mathbb{N} := \{\emptyset\} \cup \Big\{n \colon \underset{m\in\mathbb{N}}{\exists}\, n = \operatorname{succ}(m)\Big\}.$$

the Peano axioms are satisfied.
(3 Marks)

Exercise 2. Determine whether the relation R on the set of all rational numbers is reflexive, symmetric and/or transitive, where $(x, y) \in R$ if and only if

i) $x + y = 0$
ii) $x - y \in \mathbb{Z}$
iii) $xy = 0$
iv) $x = 1$ or $y = 1$
v) $x = \pm y$
vi) $x = 2y$
vii) $xy \geq 0$
viii) $x = 1$

(8 × 1 Mark)

Exercise 3. Let $\mathbb{Z}^2$ be the set of all pairs of integers. Define addition and multiplication on $\mathbb{Z}^2$ by

$$(m, n) \cdot (p, q) := (m \cdot p, n \cdot q) \qquad \text{and} \qquad (m, n) + (p, q) := (q \cdot m + p \cdot n, nq) \tag{1}$$

for $(m, n), (p, q) \in \mathbb{Z}^2$. Define the equivalence relation

$$(n, m) \sim (p, q) \qquad :\Leftrightarrow \qquad n \cdot q = m \cdot p, \tag{2}$$

giving rise to the partition $\mathbb{Z}^2/\sim$.

i) Show that the multiplication in (1) can be used to define multiplication on $\mathbb{Z}^2/\sim$ via

$$[(m, n)] \cdot [(p, q)] := [(m \cdot p, n \cdot q)] \tag{3}$$

for classes $[(m, n)], [(p, q)] \in \mathbb{Z}^2/\sim$. In particular, you need to show that this multiplication is well-defined, i.e., it doesn't depend on the representatives of the classes. In other words, if $(m, n), (m', n') \in [(m, n)]$ and $(p, q), (p', q') \in [(p, q)]$, then

$$[(m \cdot p, n \cdot q)] = [(m' \cdot p', n' \cdot q')] \qquad \text{that is} \qquad (m \cdot p, n \cdot q) \sim (m' \cdot p', n' \cdot q'). \tag{4}$$

ii) Do the same for the addition in (1).

(2 + 2 Marks)

Exercise 4. Prove the following statements by induction:

i) Let $n \in \mathbb{N} \setminus \{0\}$. Show that $133 \mid (11^{n+1} + 12^{2n-1})$.

ii) Let $n \in \mathbb{N} \setminus \{0, 1\}$. Show that $\prod_{j=1}^{n-1} \left(1 + \frac{1}{j}\right)^j = \frac{n^n}{n!}$.

iii) Let $n \in \mathbb{N}$ and $h > -1$. Show that $1 + nh \leq (1 + h)^n$.

(3 × 2 Marks)

Exercise 5. A guest at a party is *celebrity* if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party, for if there were two, they would know each other. A particular party may have no celebrity. You assignment is to find the celebrity, if one exists, at a party, by asking only one type of question – asking a guest whether they know a second guest. Everyone must answer your qestion truthfully.
Use mathematical induction to show that if there are n people at the party, then you can find the celebrity, if there is one, with at most $3(n-1)$ questions.
(3 Marks)

Exercise 6. Use strong induction to show that every $n \in \mathbb{N} \setminus \{0\}$ can be written as a sum of distinct powers of 2, i.e., as a sum of a subset of integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$ etc.
(*Hint*: For the inductive step, separately consider the case where $k+1$ is even and where it is odd. When it is even, note that $(k+1)/2 \in \mathbb{N}$.)
(3 Marks)

Exercise 7. Show that $P(n, k)$ is true for all $n, k \in \mathbb{N}$ if

i) $P(0,0)$ and $\underset{n,k \in \mathbb{N}}{\forall} P(n,k) \Rightarrow (P(n+1,k) \wedge P(n,k+1))$

ii) $\underset{k \in \mathbb{N}}{\forall} P(0,k)$ and $\underset{n \in \mathbb{N}}{\forall} P(n,k) \Rightarrow P(n+1,k)$

(2 + 2 Marks)

Exercise 8. Use Exercise 7 to prove that for all $n, k \in \mathbb{N} \setminus \{0\}$

$$\sum_{j=1}^{n} \prod_{i=0}^{k-1} (j+i) = \frac{1}{k+1} \prod_{i=0}^{k} (n+i)$$

(3 Marks)

Exercise 9. Use mathematical induction to prove the *multinomial expansion*

$$(x_1 + \cdots + x_k)^n = \sum_{n_1 + \cdots + n_k = n} \frac{n!}{n_1! \cdots n_k!} x_1^{n_1} \cdots x_k^{n_k},$$

where $k \in \mathbb{N} \setminus \{0\}$ and $n_1, \ldots, n_k \in \mathbb{N}$.
(3 Marks)

Exercise 10. Prove that the induction axiom implies the well-ordering principle.
(3 Marks)