Discrete Mathematics

Assignment 3

Date Due: 8:00 PM, Thursday, the 9th of June 2011

Office hours: Tuesdays, 1:00-3:00 PM, and Wednesdays, 12:00-1:00 PM

Exercise 1. Let $S \subset \mathbb{N}^2$ be defined by

- $(0, 0) \in S$,
- $(a, b) \in S \Rightarrow \big(((a + 2, b + 3) \in S) \wedge ((a + 3, b + 2) \in S)\big)$.

i) List the elements of S produced by the first five applications of the recursive definition.

ii) Use strong induction on the number of applications of the recursive step of the definition to show that $(a, b) \in S$ implies $5 \mid (a + b)$.

iii) Use structural induction to show that $(a, b) \in S$ implies $5 \mid (a + b)$.

(1 + 2 + 2 Marks)

Exercise 2. Prove that in a bit string, the string 01 occurs at most one more time than the string 10.
(2 Marks)

Exercise 3. The *reversal* w^R of a string w is the string consisting of the symbols of w in reverse order.

i) Find the reversal of the bit strings 0101, 1 1011 and 1000 1001 0111.

ii) Give a recursive definition of the reversal of a string. (*Hint: First define the reversal of the empty string. Then write a string w of length $n+1$ as xy, where x is a string of length n and express the reversal of w in terms of x^R and y.*)

iii) Use structural induction to prove that $(w_1 w_2)^R = w_2^R w_1^R$.

(1 + 2 + 2 Marks)

Exercise 4. The concatenation of i copies of a string w is denoted by w^i.

i) Give a recursive definition of w^i.

ii) Use induction to prove that $l(w^i) = i \cdot l(w)$.

iii) Show that $(w^R)^i = (w^i)^R$.

(2 + 2 + 2 Marks)

Exercise 5. Use structural induction to show that $n(T) \geq 2h(T) + 1$, where T is a full binary tree, $n(T)$ equals the number of vertices of T and $h(T)$ is the height of T.
(2 Marks)

Exercise 6. Let $\mathbb{Z}_+ := \{x \in \mathbb{Z} \colon x > 0\} = \mathbb{N} \setminus \{0\}$.

i) Show that the polynomial function

$$f \colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+, \qquad f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$$

is bijective.

ii) Show that

$$g \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \qquad g(m,n) = f\big((3m+1)^2, (3n+1)^2\big)$$

is injective.

Remark: It is an open question whether there is an injective polynomial function $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$.
(2+2 Marks)

Exercise 7.

i) Give the pseudocode for an algorithm that will count the bumber of 1s in a bit string by examining each bit of the string to determine whether it is a 1 bit.

ii) Give a big-O estimate for the number of comparisons used in this algorithm.

iii) Explain why the following algorithm determines the number of 1 bits in the bit string S:

procedure: *bitcount*(S: bit string)
  count := 0
  while $S \neq 0$ **do**
    count := *count* + 1
    $S := S$ AND $(S-1)$
  end while{*count* is the number of 1s in S}

Here $S-1$ is the bit string obtained by changing the rightmost 1 bit of S to a 0 and all the 0 bits to the right of this to 1s. AND denotes the bitwise conjunction.

iv) How many bitwise AND operations are needed to find the number of 1 bits in S using the above *bitcount* algorithm?

(4 × 2 Marks)

Exercise 8.

i) Give the pseudocode for an algorithm based on the linear search for determining the correct position in which to insert a new element in an already sorted list.

ii) Give the pseudocode for an algorithm based on the binary search for determining the correct position in which to insert a new element in an already sorted list.

(2 + 2 Marks)

Exercise 9. The *binary insertion sort* is a variation of the insertion sort that uses a binary search technique rather than a linear search technique to insert the i element in the correct place among the previously sorted elements.

i) Express the binary insertion sort in pseudocode.

ii) Compare the number of comparisons used by the insertion sort and the binary insertion sort to sort the list $7, 4, 3, 8, 1, 5, 4, 2$.

(2 + 2 Marks)