Discrete Mathematics

Assignment 8

Date Due: 8:00 PM, Thursday, the 14th of July 2011

Office hours: Tuesdays, 1:00-3:00 PM, and Wednesdays, 12:00-1:00 PM

Exercise 1. Let (a_n) be a sequence of real numbers. We define the sequences of *backward differences* $(\nabla^k a_n)$ as follows:

- The *first (backward) difference* is given by $\nabla a_n = a_n - a_{n-1}$.
- The *kth (backward) difference* is given by $\nabla^k a_n = \nabla^{k-1} a_n - \nabla^{k-1} a_{n-1}$.

i) Find (∇a_n) and $(\nabla^2 a_n)$ for the following sequences:

a) $a_n = 4$, b) $a_n = 2n$, c) $a_n = n^2$, d) $a_n = 2^n$.

ii) Show that $a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$. Use this to express the recurrence relation $a_n = a_{n-1} + a_{n-2}$ in terms of a_n, ∇a_n and $\nabla^2 a_n$.

iii) Prove that a_{n-k} can be expressed in terms of $a_n, \nabla a_n, \ldots, \nabla^k a_n$. Deduce that any recurrence relation for the sequence a_n can be written in terms of backward differences. The resulting equation is called a *difference equation*. Such equations occur when "discretizing" differential equations, for example, in numerical solution algorithms.

($4 \times 1 + 2 + 3$ Marks)

Exercise 2. Solve the following recurrence relations:

$$a_n = a_{n-1} + 6a_{n-2}, \qquad n \geq 2, \qquad a_0 = 3,\ a_1 = 6,$$
$$a_{n+2} = -4a_{n+1} + 5a_n, \qquad n \geq 0, \qquad a_0 = 2,\ a_1 = 8.$$

(2×2 Marks)

Exercise 3.

i) Use the solution obtained from solving the recurrence relation for the Fibonacci numbers to show that f_n is the integer closest to $\Phi^n/\sqrt{5}$, where $\Phi := (1+\sqrt{5})/2$ is the *golden ratio*.

ii) Determine the values of n for which $f_n > \Phi^n/\sqrt{5}$ and the values of n for which $f_n < \Phi^n/\sqrt{5}$.

($2 + 2$ Marks)

Exercise 4. The *Lucas numbers* are defined by

$$L_n = L_{n-1} + L_{n-2}, \qquad L_0 = 2, \qquad L_1 = 1.$$

i) Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, 4, \ldots$, where f_n is the nth Fibonacci number.

ii) Find an explicit formula for the Lucas numbers.

($2 + 2$ Marks)

Exercise 5. Prove Theorem 2.4.11 of the lecture, which states that all solution to a linear homogeneous recurrence relation of degree two are of the form

$$a_n = \alpha_1 \cdot r_0^n + \alpha_2 \cdot n r_0^n, \qquad \alpha_1, \alpha_2 \in \mathbb{R},\ n \in \mathbb{N}.$$

if there is only a single characteristic root r_0.
(3 Marks)

Exercise 6. Find all solutions of the following recurrence relations:

$$\begin{aligned} a_n &= -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n, \\ a_n &= -5a_{n-1} - 6a_{n-2} + 2^n + 3n, \\ a_n &= 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n, \end{aligned}$$

(3 × 2 Marks)

Exercise 7. In this exercise, assume that f is an increasing function satisfying the recurrence relation $f(n) = af(n/b) + cn^d$ with $a \geq 1$, $b \in \mathbb{N} \setminus \{0, 1\}$, $c, d \in \mathbb{R}_+$. Our goal is to prove the Master Theorem 2.4.22 of the lecture.

i) Show that if $a = b^d$ and n is a power of b, then $f(n) = f(1)n^d + cn^d \log_b n$.

ii) Show that if $a = b^d$, then $f(n) = O(n^d \log n)$.

iii) Show that if $a \neq b^d$ and n is a power of b, then

$$f(n) = C_1 n^d + C_2 n^{\log_b a}, \qquad C_1 = \frac{b^d c}{b^d - a}, \qquad C_2 = f(1) + \frac{b^d c}{a - b^d}.$$

iv) Show that if $a < b^d$, then $f(n) = O(n^d)$.

v) Show that if $a > b^d$, then $f(n) = O(n^{\log_b a})$.

(2 + 1 + 2 + 1 + 1 Marks)

Exercise 8. A recursive algorithm for modular exponentiation is given in Example 3 of Section 4.4, page 312 of the textbook.

i) Set up a divide-and-conquer recurrence relation for the number of modular multiplications required to compute $a^n \bmod m$, where $a, m, n \in \mathbb{Z}_+$.

ii) Construct a big-O estimate for the number of modular multiplications required to compute $a^n \bmod m$

(2 + 1 Marks)