Discrete Mathematics

Assignment 9

Date Due: 8:00 PM, Thursday, the 21st of July 2011

Office hours: Tuesdays, 1:00-3:00 PM, and Wednesdays, 12:00-1:00 PM

Exercise 1. For the following questions it is recommended that you set up a generating function for the answer and then use a computer to find the corresponding coefficient of x^n.

i) Suppose that you roll four fair twenty-sided dice. What is the probability that their sum is equal to 19?

ii) Suppose that you roll two eight-sided dice. What is the probability that their sum is eual to the expected value, 9?

iii) Suppose that you roll a six-sided and a ten-sided die. What is the probability that the sum of the results is eual to the expected value, 9?

(3 × 2 Marks)

Exercise 2. Show that the generating function for the Fibonacci numbers is given by

$$F(x) = \frac{x}{1 - x - x^2}.$$

(2 Marks)

Exercise 3. Integrate the generating function for the sequence

$$a_k = \begin{cases} 1 & 0 \le k \le n-1 \\ 0 & k \ge n \end{cases}$$

to prove that

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \int_0^1 \frac{1 - x^n}{1 - x}\, dx.$$

Then substitute $x = 1 - y$ in the integral to obtain

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k} \binom{n}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

(2 + 2 Marks)

Exercise 4. Show that for any $k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}$$

(3 Marks)

Exercise 5. You are selling tickets for the JI's end-of-term ball. Each ticket costs 5 RMB, but you start out having no money to make change.

i) There is a queue of $2n$ people waiting to buy tickets. Of these, n have only a 5 RMB bill and n have a only 10 RMB bill. What is the probability that you will be able to sell tickets to all $2n$ people and always be able to give change?

Hint/Clarification: The first customer must be one who pays with a 5 RMB bill (because you can not make change for a 10 RMB bill). The second customer can then either pay with a 5 RMB bill or with a 10 RMB bill (you can make change with the money from the first customer).

ii) Answer the same question if there are $n + m$ people, of which $n > m$ have only a 5 RMB bill and m have a only 10 RMB bill.

(2 + 4 Marks)

Exercise 6. Prove the probabilistic inclusion-exclusion principle using induction: Let S be a sample space and $A_1, \ldots, A_n \subset S$ events such that $P(A_i) \in [0, 1]$, $i = 1, \ldots, n$. Then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{1 \leq i \leq n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) \\ - + \ldots + (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

(2 Marks)

Exercise 7. In a music class, three students play violin, three students play piano and three students play the flute; two students play both piano and flute, two students play both violin and piano and two students play both violin and flute; one student plays all three instruments. How many students are in the class?
(2 Marks)

Exercise 8.

i) Give an example to show that the transitive closure of the symmetric closure of a relation is not necessarily the same as the symmetric closure of the transitive closure of this relation.

ii) Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive ' closure of this relation.

(2 + 2 Marks)

Exercise 9. Let M be a set and R a relation on M. Prove or disprove the following statements:

i) If R is reflexive, then R^2 is reflexive.

ii) If R is symmetric, then R^2 is symmetric.

iii) If R is antisymmetric, then R^2 is antisymmetric.

iv) If R is transitive, then R^2 is transitive.

(4 × 1 Marks)

Exercise 10. Consider the following relation on the set $\{1, 2, 3, 4, 5\}$:

$$R = \big\{(1, 3), (2, 4), (3, 1), (3, 5), (4, 2), (5, 1), (5, 2), (5, 4)\big\}$$

i) Draw the graph representing R and give the zero-one matrix representing R.

ii) Find the symmetric and reflexive closures of R and draw their graphs.

iii) Find the transitive closure of R and draw its graph.

(2 + 2 + 3 Marks)