Ve401 Probabilistic Methods in Engineering

Summer 2017 — Assigment 4

Date Due: 12:10 PM, Wednesday, the 21st of June 2017

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This assignment has a total of **(34 Marks)**.

Exercise 4.1

The joint density $f_{XY}\colon \Omega \to \mathbb{R}$ for the discrete bivariate random variable $(X, Y)\colon S \to \Omega = \{(x, y)\colon 1 \le x \le y \le n\}$, where $n \in \mathbb{N}$, $n \ge 1$, is given by

$$f_{XY}(x, y) = \frac{2}{n(n+1)}.$$

i) Verify that f_{XY} is in fact a density.
(1 Mark)

ii) Find the marginal densities for X and Y.
(1 Mark)

iii) Are X and Y independent?
(1 Mark)

iv) Assume that $n = 5$. Use the joint density to find $P[X \le 3 \text{ and } Y \le 2]$. Find $P[X \le 3]$ and $P[Y \le 2]$.
(2 Marks)

Exercise 4.2 The Sum of Two Continuous Random Variables

Let X and Y be continuous random variables with parameters with joint density f_{XY}. Let $U = X + Y$ and prove that the density of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{XY}(u - v, v)\, dv.$$

Hint: Consider the transformation $(x, y) \mapsto (x + y, y)$.
(2 Marks)

Exercise 4.3 The Sum of Two Exponential Distributions

Let X and Y be independent exponentially distributed random variables with parameters $\beta_1 = 3$ and $\beta_2 = 1$, respectively. Let $U = X + Y$ and show that

$$f_U(u) = \begin{cases} (e^{-u/3} - e^{-u})/2 & u > 0 \\ 0 & u \le 0 \end{cases}$$

(2 Marks)

Exercise 4.4 The Sum of Two Normal Distributions

Let X_1 and X_2 be normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Show that the linear combination

$$Y = \lambda_1 X_1 + \lambda_2 X_2$$

follows a normal distribution and find the mean and variance of Y.
(4 Marks)

Exercise 4.5

Let $X = (X_1, X_2)$ be a random vector. Then we define the expectation vector and the variance-covariance matrix as follows:

$$\mathrm{E}[X] := \begin{pmatrix} \mathrm{E}[X_1] \\ \mathrm{E}[X_2] \end{pmatrix}, \qquad \operatorname{Var} X := \begin{pmatrix} \operatorname{Var} X_1 & \operatorname{Cov}(X_1, X_2) \\ \operatorname{Cov}(X_2, X_1) & \operatorname{Var} X_2 \end{pmatrix}.$$

Let A be a constant 2×2 matrix and $Y = (Y_1, Y_2) = AX$.

i) Show that $\mathrm{E}[AX] = A\,\mathrm{E}[X]$.
(1 Mark)

ii) Show that $\mathrm{Var}(AX) = A(\mathrm{Var}\,X)A^T$.
(2 Marks)

iii) Suppose that X_1 and X_2 follow independent normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that the joint density is given by

$$f_X(x) = f_X(x_1, x_2) = \frac{1}{2\pi\sqrt{\det \Sigma_X}} e^{-\frac{1}{2}\langle x-\mu_X, \Sigma_X^{-1}(x-\mu_X)\rangle}$$

where $\mu_X = (\mu_1, \mu_2)$ and $\Sigma_X = \mathrm{diag}(\sigma_1^2, \sigma_2^2)$ is the 2×2 matrix with the variances on the diagonal and all other entries vanishing.
(1 Mark)

iv) Suppose that X_1 and X_2 follow independent normal distributions with means $\mu_1, \mu_2 \in \mathbb{R}$ and variances $\sigma_1^2, \sigma_2^2 > 0$, respectively. Let $Y = AX$ where A is an invertible $n \times n$ matrix. Show that

$$f_Y(y) = \frac{1}{2\pi\sqrt{|\det \Sigma_Y|}} e^{-\frac{1}{2}\langle y-\mu_Y, \Sigma_Y^{-1}(y-\mu_Y)\rangle} \qquad (*)$$

where $\mu_Y = \mathrm{E}[Y]$, $\Sigma_Y = \mathrm{Var}\,Y$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the euclidean scalar product in $\mathbb{R}^2$.
(2 Marks)

v) Show that $(*)$ can be written as

$$f_Y(y_1, y_2) = \frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\varrho^2}}\, e^{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\varrho\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right) + \left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]} \qquad (**)$$

where μ_{Y_i} is the mean and $\sigma_{Y_i}^2$ the variance of Y_i, $i = 1, 2$, and ϱ is the correllation coeffcient of Y_1 and Y_2.
(2 Marks)

Remark: The above statements (except v), of course) generalize to n-dimensional random vectors $(X_1, \ldots, X_n)$.

Exercise 4.6

Let $((X_1, X_2), f_{X_1X_2})$ be a continuous bivariate random variable[1] following the *bivariate normal distribution* given by

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\varrho^2}}\, e^{-\frac{1}{2(1-\varrho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\varrho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

with parameters $\sigma_1, \sigma_2 > 0$, $\mu_1, \mu_2 \in \mathbb{R}$ and $|\varrho| < 1$.

i) Verify that the marginal density for X_1 is that of a normal distribution with mean μ_1 and variance σ_1^2.
(3 Marks)

ii) Show that ϱ is the coefficient of correlation between X_1 and X_2.
(3 Marks)

iii) Show that X_1 and X_2 are independent if and only if $\varrho = 0$. Is this propery true for a bivariate random variable with an arbitrary distribution? Why or why not?
(2 Marks)

iv) Prove that

$$\mu_{X_2|x_1} = \mu_2 + \varrho\frac{\sigma_2}{\sigma_1}(x - \mu_1).$$

where $X_1 \mid x_2$ is the conditional random variable X_1 in the case $X_2 = x_2$.
(3 Marks)

v) The life X_1 of a tube and the filament diameter X_2 are distributed as a bivariate normal random variable with the parameters $\mu_1 = 2000$ hours, $\mu_2 = 0.1$ inch, $\sigma_1^2 = 2500\,\text{hours}^2$, $\sigma_2^2 = 0.01\,\text{inch}^2$ and $\varrho = 0.87$.

The quality-control manager wishes to determine the life of each tube by measuring the filament diameter. If a filament diameter is 0.098 inch, what is the probability that the tube will last 1950 hours or longer?
(2 Marks)

[1] This exercise was part of the first midterm exam in the fall term of 2008. You should not be afraid of evaluating integrals!