Ve401 Probabilistic Methods in Engineering

Summer 2017 — Assigment 5

Date Due: 12:10 PM, Monday, the 3rd of July 2017

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This assignment has a total of **(45 Marks)**.

Exercise 5.1
The shear strengths of 100 spot welds in a titanium alloy follow:

5408 5431 5475 5442 5376 5388 5459 5422 5416 5435 5420 5429 5401 5446 5487 5416 5382 5357 5388
5457 5407 5469 5416 5377 5454 5375 5409 5459 5445 5429 5463 5408 5481 5453 5422 5354 5421 5406
5444 5466 5399 5391 5477 5447 5329 5473 5423 5441 5412 5384 5445 5436 5454 5453 5428 5418 5465
5427 5421 5396 5381 5425 5388 5388 5378 5481 5387 5440 5482 5406 5401 5411 5399 5431 5440 5413
5406 5342 5452 5420 5458 5485 5431 5416 5431 5390 5399 5435 5387 5462 5383 5401 5407 5385 5440
5422 5448 5366 5430 5418

i) Construct a stem-and-leaf diagram for the weld strength data and comment on any important features that you notice.
(1 Mark)

ii) Construct a histogram. Comment on the shape of the histogram. Does it convey the same information as the stem-and-leaf display?
(2 Marks)

iii) Construct a box plot of the data and write an interpretation of the plot. How does the box plot compare in interpretive value to the original stem-and-leaf diagram?
(2 Marks)

Exercise 5.2
Temperature differences between the warm upper surface of the ocean and the colder deeper levels can be utilized to convert thermal energy to mechanical energy. This mechanical energy can in turn be used to to poduce electrical power using a vapor turbine.
Let X denote the difference in temperature between the surface of the water and the water at a depth of one kilometer. Measurements are taken at 15 randomly selected sites in the Gulf of Mexico. These data result in the following temperatures.

22.5 23.8 23.2 22.8 10.1* 23.5 24.0 23.2 24.2 24.3 23.3 23.4 23.0 23.5 22.8

i) Construct a double stem-and-leaf diagram for these data.
(2 Marks)

ii) Find the sample mean, sample median and sample standard deviation for these data.
(1 Mark)

iii) Note that the starred observation in the data set is very different from the others. It is a potential outlier. Construct a boxplot for these data to verify that the value 10.1 does, in fact, qualify as an outlier.
(2 Marks)

iv) To see the effect of this outlier, drop it from the data set and calculate the sample mean, median and standard deviation for the remaining 14 observations. Comment on the changes in these quantities.
(2 Marks)

Exercise 5.3
Let W be an exponential random variable with parameter β unknown.

i) Find the method-of-moments estimator for β based on a sample size n.
(1 Mark)

ii) Find the maximum-likelihood estimator for β based on a sample size n.
(1 Mark)

Exercise 5.4

A new material is being tested for possible use in the brake shoes of automobiles. These shoes are expected to last for at least 75,000 miles. Fifteen sets of four of these experimental shoes are subjected to accelerated life testing. The random variable X, the number of shoes in each group of 4 that fail early, is assumed to be binomially distributed with $n = 4$ and p unknown.

i) Find the maximum-likelihood estimate for p based on these data:

1	0	1	0	2	1	0	1
0	1	0	0	0	1	0	

(2 Marks)

ii) If an early failure rate in excess of 10% is unacceptable from a business point of view, would you have some doubts concerning the use of this new material? Explain.
(1 Mark)

Exercise 5.5

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a random variable with variance σ^2. We have seen that the sample variance

$$S_{n-1}^2 := \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2$$

is an unbiased estimator for σ^2. It can be shown that

$$\mathrm{Var}(S_{n-1}^2) = \mathrm{MSE}(S_{n-1}^2) = \frac{1}{n}\left(\mathrm{E}[(X-\overline{X})^4] - \frac{n-3}{n-1}\sigma^4\right) = \frac{1}{n}\left(\gamma_2 + \frac{2n}{n-1}\right)\sigma^4$$

where $\gamma_2 := \mathrm{E}[(X-\mu)^4]/\sigma^4 - 3$ is called the *excess kurtosis* of a distribution. (You do not have to perform this tedious calculation!)

i) Show that if X follows a normal distribution with mean μ and variance σ^2,

$$\mathrm{MSE}(S_{n-1}^2) = \frac{2}{n-1}\sigma^4$$

(2 Marks)

ii) For $a > 0$ set

$$S_a^2 := \frac{n-1}{a}S_{n-1}^2.$$

Find $\mathrm{MSE}(S_a^2)$ and show that the mean square error is minimized for

$$a = n + 1 + \frac{n-1}{n}\gamma_2$$

In the case of a normal distribution with mean μ and variance σ^2, show that this reduces to $a = n + 1$. Conclude that a biased estimator may be "better" overall than an unbiased estimator.
(3 Marks)

Exercise 5.6

Let $X_1, \ldots, X_n$ be a random sample of size n from a normal distribution with variance σ^2 and let S^2 be the sample variance. It can be shown that $(n-1)S^2/\sigma^2$ follows a chi-squared distribution with $n - 1$ degrees of freedom.

Is the sample standard deviation S an unbiased estimator for the population standard deviation σ? If not, find a coefficient c_n so that $c_n S$ is unbiased for σ.
(3 Marks)

Exercise 5.7

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a uniform continuous random variable[1] on the interval $[0, \theta]$, $\theta > 0$, i.e., having the density

$$f(x) = \begin{cases} 1/\theta, & x \in [0, \theta], \\ 0 & \text{otherwise.} \end{cases}$$

i) Show that the method-of-moments estimator for θ is

$$\hat{\theta}_{\mathrm{MOM}} = 2\overline{X}$$

and verify that its means square error is

$$\mathrm{MSE}(\hat{\theta}_{\mathrm{MOM}}) = \frac{\theta^2}{3n}$$

(2 Marks)

ii) Show that the maximum-likelihood estimator for θ is

$$\hat{\theta}_{\mathrm{ML}} = \max_{1 \le k \le n} X_k.$$

(2 Marks)

iii) Find a formula for the cumulative distribution function $P[\hat{\theta}_{\mathrm{ML}} \le x] = P[X_1 \le x, \dots, X_n \le x]$ and then differentiate to obtain the density

$$f_{\hat{\theta}_{\mathrm{ML}}}(x) = \begin{cases} nx^{n-1}/\theta^n, & 0 < x < \theta, \\ 0 & \text{otherwise.} \end{cases}$$

(2 Marks)

iv) Find an upper one-sided $100(1-\alpha)\%$ confidence interval for θ, i.e., find $L > \hat{\theta}_{\mathrm{ML}}$ such that $\theta \in [\hat{\theta}_{\mathrm{ML}}, L]$ with $100(1-\alpha)\%$ probability.
(2 Marks)

v) Find the bias and the variance of $\hat{\theta}_{\mathrm{ML}}$. Is $\hat{\theta}_{\mathrm{ML}}$ unbiased? How do the bias and the variance behave as $n \to \infty$? Verify that the mean square error is given by

$$\mathrm{MSE}(\hat{\theta}_{\mathrm{ML}}) = \frac{2\theta^2}{(n+2)(n+1)}.$$

Compare with the mean square error of $\hat{\theta}_{\mathrm{MOM}}$.
(4 Marks)

vi) Show that the estimator

$$\hat{\theta}^* := \frac{n+2}{n+1} \max_{1 \le k \le n} X_k$$

has a smaller mean square error than $\hat{\theta}_{\mathrm{ML}}$ whenever $n \ge 1$. Conclude that the method of maximum likelihood does not always yield the "best" estimator.
(2 Marks)

Exercise 5.8

Suppose[2] that the random variable X has the probability density

$$f(x) = \begin{cases} (\gamma+1)x^\gamma, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $X_1, X_2, \dots, X_n$ be a random sample of size n. Find the maximum likelihood estimator for γ.
(2 Marks)

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[1] This exercise is adapted from Example 5 of the very readable discussion of maximum likelihood estimators in L. Le Cam, *Maximum Likelihood: An Introduction*, International Statistical Review / Revue Internationale de Statistique, Vol. 58, No. 2 (Aug., 1990), pp. 153-171, http://www.jstor.org/stable/1403464

[2] This exercise is taken from the second midterm exam of Fall 2008.

Exercise 5.9
Consider a general two-sided $100(1-\alpha)\%$ confidence interval for the mean μ when σ is known:

$$\overline{x} - z_{\alpha_1}\sigma/\sqrt{n} \leq \mu \leq \overline{x} - z_{\alpha_2}\sigma/\sqrt{n}$$

where $\alpha_1 + \alpha_2 = \alpha$. Show that the length of the interval, $\sigma(z_{\alpha_1} + z_{\alpha_2})/\sqrt{n}$ is minimized when $\alpha_1 = \alpha_2 = \alpha/2$.
(2 Marks)

Exercise 5.10
Let $X = (X_1, X_2)$ where X_1 and X_2 follow i.i.d. normal distributions with variance σ^2 and mean μ. Let $Y = (Y_1, Y_2) = AX$ with $A^T = A^{-1}$. Show that Y_1 and Y_2 are independent with mean $\mathrm{E}[Y] = A\,\mathrm{E}[X]$ and variance $\mathrm{Var}\, Y = \sigma^2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. (See also Exercise 4.5 for the notation used here.)
(2 Marks)