Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 1

Date Due: 2:00 PM, Thursday, the 8th of October 2015

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Exercise 1. The differential equation[1] for the free fall from a large distance is $\ddot{s} = -GMs^{-2}$. Let s_1 be the solution given by

$$s_1(t) = \alpha t^{2/3}, \qquad \text{with} \quad \alpha = (9GM/2)^{1/3}, \quad t > 0.$$

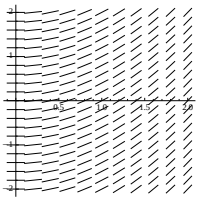
We have seen in the lecture that this solution corresponds to a particle escaping from the surface of the Earth with initial velocity 11.2 km/s. The obvious question is whether there may exist another solution of $\ddot{s} = -GMs^{-2}$ that also escapes from earth but has a lower initial velocity. This exercise shows that this is not the case, i.e., 11.2 km/s is the smallest velocity which allows a particle to escape from Earth.

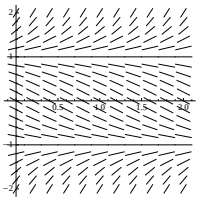
Let s be any positive solution of $\ddot{s} = -GMs^{-2}$ in an interval $[t_0, t_1)$ with $t_0 > 0$ and $t_0 < t_1 \leq \infty$ such that $s(t_0) = s_1(t_0)$ and $0 < \dot{s}(t_0) < \dot{s}_1(t_0)$. Prove that for $t \in (t_0, t_1)$ the following holds: $s(t) < s_1(t)$ and $\dot{s}(t) < \dot{s}_1(t)$. Furthermore, show that s is bounded above and $\dot{s}$ has one zero, i.e., s describes a return trajectory.

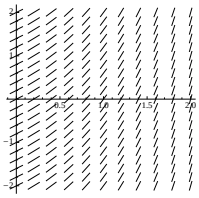
Hint: Derive a differential equation for the difference $d(t) = s_1(t) - s(t)$ and conclude from it that $\dot{d}$ is increasing as long as d is positive. Note that $\ddot{s} < 0$ and $\dot{s}_1 \to 0$ as $t \to \infty$.
(6 Marks)

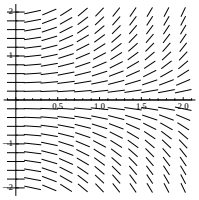
Exercise 2. Match the following differential equations to the corresponding direction field:

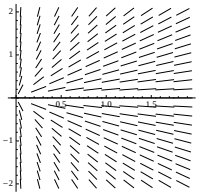
1) $y' = y^2 - 1$ 2) $y' = \dfrac{y}{x}$ 3) $y' = xy$ 4) $y' = e^x$ 5) $y' = xy^2$ 6) $y' = x$

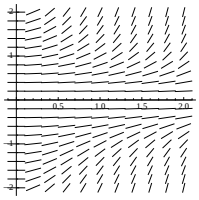
a) 

b) 

c) 

d) 

e) 

f) 

(6 × 1 Mark)

Exercise 3. Sketch the direction field for the following differential equations, and sketch the solution that passes through the point $(1, 1)$:

a) $y' = \sin(\pi xy)$, b) $y' = \dfrac{1}{1 + y^2}$.

(2+2 Marks)

Exercise 4. In the 1950 excavation[2] at Nippur, a city of Babylonia, charcoal from a roof beam gave an average count in 1950 of 4.09 disintegrations per minute per gram. Living wood gave 6.68 disintegrations. Assuming that this charcoal was fomred during the time of Hammurabi's reign, find an estimate for the likely time of Hammurabi's succession.
(3 Marks)

[1] *Walter*, Introduction, Exercise
[2] *Braun*, Ch. 1.3, Ex. 8

Exercise 5. This exercise[3] refers to Chapter 1.3 of *Braun*, which will also be discussed in the recitation class. Read the discussion of the art forgeries in this chapter, and then demonstrate that the paintings "Washing of Feet," "Woman Reading Music" and "Woman playing Mandolin" are forgeries.
(3 Marks)

Exercise 6. Plot the direction field of the equation $xy' = y^2 - y$. Then find, if possible, at least one solution that passed through each of the indicated points:

a) $(2, 1/4)$, b) $(1/2, 1/2)$, c) $(0, 2)$, d) $(0, 1)$.

(Do not worry about proving uniqueness; stating one solution in each case is sufficient.)
(4 Marks)

Exercise 7. Find the general solution to the following equations:

a) $y' = (1 + x)(1 + y)$, b) $y' = e^{x+y+3}$.

(2+2 Marks)

[3] *Braun*, Ch. 1.3, Ex. 2-4