Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 2

Date Due: 10:00 AM, Thursday, the 8th of October 2015

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Suppose that a grams of chemical A are combined with b grams of chemical B to give chemical C. If there are M parts of A and N parts of B in the compund C (i.e., the chemical reaction equation is $M\,A + N\,B \to C$) and $X(t)$ is the number of grams of chemical C formed, then the number of grams of chemical A and the number of grams of chemical B remaining at time t are, respectively,

$$a - \frac{M}{M+N}X \qquad \text{and} \qquad b - \frac{N}{M+N}X.$$

The law of mass action states that when no temperature change is involved, the rate at which the two substances react is proportional to the product of the amount of A and B that remain at time T:

$$\frac{dX}{dt} \sim \left(a - \frac{M}{M+N}X\right)\left(b - \frac{N}{M+N}X\right)$$

With a constant of proportionality $k > 0$ we obtain

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$$

where $\alpha = a(M+N)/M$ and $\beta = b(M+N)/N$. A reaction governed by this law is said to be a *second-order reation.*

Exercise 1. Two chemicals A and B are combind to form a chemical C in a second-order reaction. Initially ther are 40 grams of A and 50 grams of B and for each gram of B, 2 grams of A are used. It is observed that 10 grams of C are formed in 5 minutes.

i) How much is formed in 20 minutes?

ii) What is the limiting amount of C as time $t \to \infty$?

iii) How much of chemicals A and B remain as $t \to \infty$?

(1 + 1 + 2 Marks)

Exercise 2. Solve the initial value problem

$$\dot{x} = \frac{2}{t^2} - x^2, \qquad x(1) = 2.$$

(2 Marks)

Exercise 3. A population grows according to the logistic law, with a limiting population of $5 \cdot 10^8$ individuals. When the population is low it doubles every 40 minutes. What will the population be after two hours if initially it is (a) 10^8, (b) 10^9?
(2 + 2 Marks)

Exercise 4. The following equation describes a "constant harvesting" model,

$$y' = (5 - 4y)y - 1.$$

i) Find the general solution.

ii) What is carrying capacity?

iii) What is the threshold population?

iv) Use Mathematica to graph one solution that leads to extinction and another solution that converges to the carrying capacity.

(2 + 1 + 1 + 2 Marks)

Exercise 5. We can model a population which becomes susceptible to epidemics in the following manner. Assume that our population is originally governed by the logistic law

$$\frac{dp}{dt} = ap - bp^2, \qquad a, b > 0, \tag{1}$$

and that an epidemic strikes as soon as p reaches a certain value Q, with Q less than the limiting population a/b. At this stage the vital coefficients become $A < Q$, $B < b$, and (1) is replaced by

$$\frac{dp}{dt} = Ap - Bp^2, \qquad A, B > 0, \tag{2}$$

Suppose that $Q > A/B$. The population then starts decreasing. A point is reached when the population falls below a certain value $q > A/B$. At this moment the epidemic ceases and the population again begins to grow following (1), until the incidence of a fresh epidemic. In this way there are periodic fluctuations of p between q and Q. We now indicate how to calculate the period T of these fluctuations.

i) Show that the time T_l taken by the first part of the cycle, when p increases from q to Q is given by

$$T_1 = \frac{1}{a} \ln\left(\frac{Q(a - bq)}{q(a - bQ)}\right).$$

ii) Show that the time T_2 taken by the second part of the cycle, when p decreases from Q to q is given by

$$T_2 = \frac{1}{A} \ln\left(\frac{q(QB - A)}{Q(qB - A)}\right).$$

Thus, the time for the entire cycle is $T_1 + T_2$.

It has been observed that plagues appear in mice populations whenever the population becomes too large. Further, a local increase of density attracts predators in large numbers. These two factors will succeed in destroying 97-98% of a population of small rodents in two or three weeks, and the density then falls to a level at which the disease cannot spread. The population, reduced to 2% of its maximum, finds its refuges from the predators sufficient, and its food abundant. The population therefore begins to grow again until it reaches a level favorable to another wave of disease and predation. Now, the speed of reproduction in mice is so great that we may set $b = 0$ in (1). In the second part of the cycle, on the contrary, A is very small in comparison with B, and it may be neglected therefore in (2).

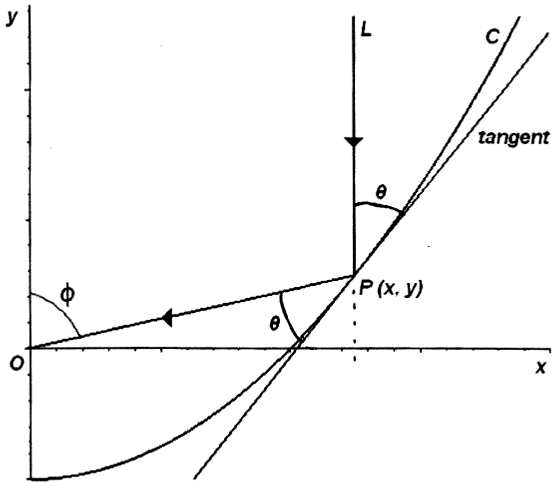
iii) Under these assumptions, show that

$$T_1 = \frac{1}{a} \ln\left(\frac{Q}{q}\right) \qquad \text{and} \qquad T_2 = \frac{Q - q}{qQB}.$$

Assuming that T_1 is approximately four years, and Q/q is approximately fifty, show that a is approximately one. This value of a, incidentally, corresponds very well with the rate of multiplication of mice in natural circumstances.

(2 + 2 + 2 Marks)

Exercise 6. Light strikes a plane curve C in such a manner that all beams L parallel to the y-axis are reflected to a single point O (see the diagram below). The objective of this exercise is to determine the differential equation for the fucntion $y = f(x)$ describing the curve C.



i) Show geometrically that $\phi = 2\theta$, $\tan\phi = \frac{x}{y}$ and $\tan(\pi/2 - \theta) = \frac{dy}{dx}$.

ii) Use the identity $\tan(\pi/2 - x) = \frac{1}{\tan x}$ to show that $\tan\theta = \frac{dx}{dy}$.

iii) Use the identity $\tan(2x) = 2\tan x/(1 - \tan^2 x)$ to derive the ODE

$$x\left(\frac{dx}{dy}\right)^2 + 2y\frac{dx}{dy} = x. \qquad (*)$$

iv) Substitute $w = x^2$ in $(*)$ to obtain a differential equation of Clairaut type. Solve this equation, and resubstitute $w = x^2$ to obtain a solution of $(*)$.

v) What do the above calculations tell you about the types of curves that can be used to focus rays into a single point?

$(3 \times \frac{1}{2} + 1 \times \frac{1}{2} + 2 + 2 + 2$ Marks)