Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 4

Date Due: 10:00 AM, Thursday, the 22nd of October 2015

Exercise 4.1. Determine the eigenvalues, eigenvectors and eigenspaces for the following matrices:

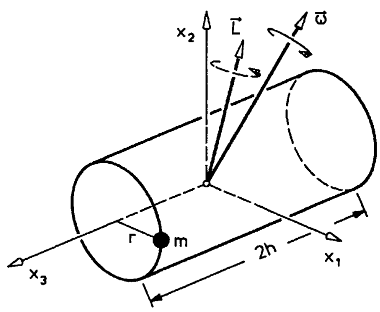
$$A = \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

(2 + 2 Marks)

Exercise 4.2. Let $A \in \operatorname{Mat}(n \times n; R)$ be symmetric ($A = A^T$) and $Q_A\colon \mathbb{R}^n \to \mathbb{R}$, $Q_A(x) = \langle x, Ax\rangle$ the associated quadratic form. Show that the maximum (resp. minimum) value of Q_A when restricted to the unit sphere is given by the largest (resp. smallest) eigenvalue of A.

Hint: Use Lagrange multipliers for finding the extremum under the constraint $|x|^2 = \langle x, x\rangle = 1$.
(2 Marks)

Exercise 4.3. A cylindrical flywheel ($r = h = 30\,\text{cm}$, mass $M = 1\,\text{kg}$) has a point-mass of $m = 0.1\,\text{kg}$ attached at its edge. In the sketched coordinate system (fixed to the cylinder) the *inertial tensor* has the form



$$I = \begin{pmatrix} \frac{M}{12}(3r^2+4h^2)+mh^2 & 0 & -mrh \\ 0 & \frac{M}{12}(3r^2+4h^2)+m(h^2+r^2) & 0 \\ -mrh & 0 & \frac{M}{2}r^2+mr^2 \end{pmatrix}$$

If the rotational velocity of the flywheel is $\vec{\omega}$, then the *rotational energy* is the quadratic form

$$T = \frac{1}{2}\langle\vec{\omega}, I\vec{\omega}\rangle$$

and the *angular momentum* is $\vec{L} = I\vec{\omega}$. If the flywheel can rotate freely in space, $\vec{L}$ remains fixed and $\vec{\omega}$ rotates about $\vec{L}$ (*nutation*).

i) Calculate the numerical value of I as well as of $\vec{L}$ and T when $\vec{\omega} = \vec{e}_3$.

ii) Using the above numerical values, find the *principal moments of inertia* (eigenvalues of I) and the *principal axes of inertia* (eigenvectors of I). For which axes $\vec{\omega}$ with $|\vec{\omega}| = 1$ is T maximal and minimal (see Exercise 4.2 above)? Comment on the nutation for these axes.

(2 + 3 Marks)

Exercise 4.4. The matrix

$$A = \begin{pmatrix} -1 & -18 & -7 \\ 1 & -13 & -4 \\ -1 & 25 & 8 \end{pmatrix}$$

has the single eigenalue $\lambda = -2$. Find a basis of generalized eigenvectors and derive the Jordan normal form of A without explicitly multiplying $U^{-1}AU$.
(3 Marks)

Exercise 4.5. The matrix

$$A = \begin{pmatrix} 4 & -4 & -11 & 11 \\ 3 & -12 & -42 & 42 \\ -2 & 12 & 37 & -34 \\ -1 & 7 & 20 & -17 \end{pmatrix}$$

has the single eigenalue $\lambda = 3$. Find a basis of generalized eigenvectors and derive the Jordan normal form of A without explicitly multiplying $U^{-1}AU$.
(3 Marks)

Exercise 4.6. Let $A \in \text{Mat}(n \times n, \mathbb{C})$ be any matrix and $\lambda_1, \dots, \lambda_n$ the eigenvalues of A, counted with multiplicities. Show that

$$\det A = \prod_{i=1}^{n} \lambda_i, \qquad \operatorname{tr} A = \sum_{i=1}^{n} \lambda_i.$$

Furthermore, show that

$$\det(e^A) = e^{\operatorname{tr} A}.$$

(Hint: transform A to Jordan normal form and use the properties of the trace and determinant.)
(1 + 1 + 2 Marks)

Exercise 4.7. A particle of mass $m = 1$ travels in $\mathbb{R}^2$ under the influence of a constant linear force field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$. Using Newton's second law to derive a differential equation for the position x and velocity v, verify that

$$\begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ F & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}, \qquad \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{pmatrix}$$

where $F \in \text{Mat}(2, \mathbb{R})$. Find the general solution for the system of equations when $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(4 Marks)

Exercise 4.8. We consider a linear chain of springs consisting of three mass points $m > 0$ that are linked by springs with spring constants $k > 0$. We denote the equilibirum positions of the mass points by r_1, r_2, r_3 and the positions at time by $x_1(t), x_2(t), x_3(t)$, respectively. Hence $d_j(t) := x_j(t) - r_j$ describes the displacement of the jth mass point at time t.

i) Use the laws of Newton and Hooke to derive the differential equation

$$\ddot{d} = Ad, \qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \qquad A = \frac{k}{m} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \qquad (**)$$

ii) Transform $(**)$ into an equivalent first-order system $\binom{\dot{v}}{\dot{d}} = B\binom{v}{d}$ as in Exercise 4.7.

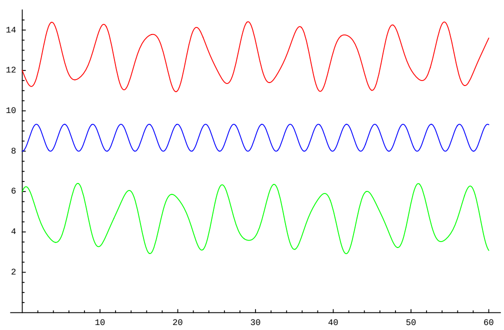
iii) For simplicity, set $k = m = 1$. Use Mathematica to find the eigenvalues, eigenvectors and the Jordan normal form J of B. Also obtain from Mathematica the matrix S such that $B = SJS^{-1}$.

iv) Obtain $\Phi(t) = e^{Bt} = Se^{Jt}S^{-1}$ from Mathematica. Verify that $\Phi(0) = \mathbb{1}$.

v) Extract the 6 linearly independent fundamental solutions $(d(t), v(t))$ und plot 6 pairs of graphs as follows: Plot the three curves $d_1(t), d_2(t), d_3(t)$ using different colors in the same graph. Then plot $v_1(t), v_2(t), v_3(t)$ together in a second graph. Do this for all six fundamental solutions.

vi) Let $r_1 = 4$, $r_2 = 8$ and $r_3 = 12$. Find the solution $(d(t), v(t))$ corresponding to the initial conditions $v(0) = (1, 0, -1)$ and $d(0) = (2, 0, 0)$ and plot the solution curves $x_1(t), x_2(t), x_3(t)$ together in a single graph, using different colors for each graph.

Solution: The graphs should look like this:



Note the irregular behaviour of the oscillations!

(2 + 1 + 3 + 2 + 4 + 3 Marks)