Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 5

Date Due: 10:00 AM, Thursday, the 29th of October 2015

Exercise 5.1. Show that the Ricatti differential equation

$$y' + g(x)y + h(x)y^2 = k(x) \qquad \text{on an open interval } I \subset \mathbb{R}$$

with $g, h \in C(I)$, $h \in C^1(I)$, $h \neq 0$ on I, can be transformed into the linear differential equation of second order,

$$u'' + \left(g - \frac{h'}{h}\right)u' - khu = 0,$$

using the transformation

$$u(x) = e^{\int h(x)y(x)\,dx}.$$

Furtermore, show that a positive solution u of the second-order ODE produces a solution $y = (\ln u)'/h$ of the Ricatti equation. Use thsi relationship to solve the initial value problem

$$y' - y + e^x y^2 = -5e^{-x}, \qquad y(0) = \eta.$$

(2 + 1 + 3 Marks)

Exercise 5.2. Find the solution to the initial value problem[1]

$$y''' + y' = \sec t \tan t, \qquad y''(0) = y'(0) = y(0) = 0,$$

by converting the third-order ODE into a system of equations and finding the solution to the system.
(3 Marks)

Exercise 5.3. Show that the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -b^2/(4a^2) & -b/a \end{pmatrix}.$$

is not diagonalizable for any values of $a, b \in \mathbb{R}$.
(2 Marks)

Exercise 5.4. Use the method of reduction of order to find the general solution to the following differential equations.[2]

$$y'' - \frac{2(t+1)}{t^2+2t-1}y' + \frac{2}{t^2+2t-1}y = 0, \qquad y_1(t) = t+1,$$
$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0, \qquad y_1(t) = \frac{\sin t}{\sqrt{t}}.$$

(2 × 2 Marks)

Exercise 5.5. A small object of mass 1 kg is attached to a spring with spring constant 1 N/m and is immersed in a viscous medium with damping constant 2 Ns/m. At time $t = 0$ the mass is lowered 1/4 m and given an initial velocity of 1 m/s in the upward direction. Show that the mass will overshoot its equilibrium position once, and then creep back to equilibrium.
(3 Marks)

Exercise 5.6. A small object of mass 4 kg is attached to an elastic spring with spring constant 64 N/m, and is acted upon by an external force $F(t) = A\cos^3(\omega t)$, $A > 0$. Find all values of ω for which resonance occurs.
(3 Marks)

[1] *Braun*, Section 3.12, Ex. 8
[2] *Braun*, Section 2.2, Ex. 10ff

Exercise 5.7. The gun of a U.S. M60 tank is attached to a spring-mass-dashpot system (i.e., a damped spring-mass system) with spring constant $100\alpha^2$ and damping constant 200α, in their appropriate units. The mass of the gun is 150 kg. Assume that the displacement $y(t)$ of the gun from its rest position after being fired at time $t = 0$ satisfies the initial value problem

$$150y'' + 200\alpha y' + 100\alpha^2 y = 0, \qquad y(0) = 0, \qquad y'(0) = 100\,\mathrm{m}\,/\,\mathrm{s}\,.$$

It is desired that one second later, the quantity $y^2 + (y')^2$ be less than 0.01. How large (numerical value) must α be to guarantee that this is so? (The spring-mass-dashpot mechanism in the M60 tanks supplied by the U.S. to Israel are critically damped, for this situation is preferable in desert warfare were one has to fire again as quickly as possible.)[3]
(4 Marks)

[3] *Braun*, Section 2.6. Ex. 7; some numerical values have been changed.