Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 6

Date Due: 10:00 AM, Thursday, the 5th of November 2015

Exercise 6.1. Let $\Omega \subset \mathbb{R}^2$ be a connected, open set. A function $u \in C^2(\Omega)$ such that $\Delta u = 0$ (where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator) is called *harmonic*.

i) Show that if f is a holomorphic function given by $f(x+iy) = u(x,y) + v(x,y)i$, where $u, v \colon \Omega \to \mathbb{R}$ are real functions, then u and v are harmonic.

ii) If u is harmonic, the Cauchy-Riemann differential equations define a harmonic function v such that $f(x+iy) := u(x,y) + v(x,y)i$ is holomorphic. This function v is called the *harmonic conjugate* of u. Find a harmonic conjugate to the function $u(x,y) = x^3 - 3xy^2$.

(2 + 2 Marks)

Exercise 6.2. Show by direct integration that if $|a| < r < |b|$, then

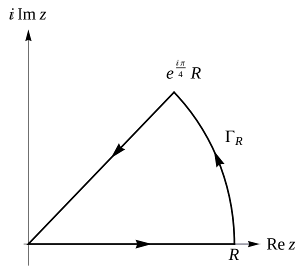
$$\oint_\gamma \frac{1}{(z-a)(z-b)} = \frac{2\pi i}{a-b}$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.
(1 Mark)

Exercise 6.3. Prove that

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$$

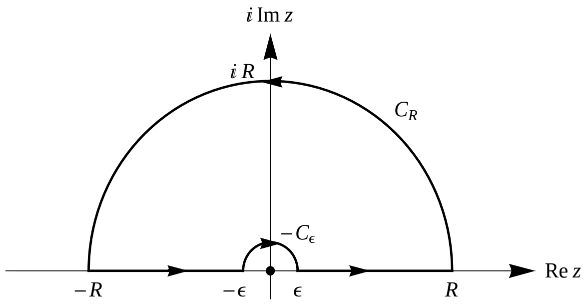
by integrating along the toy contour Γ_R (a *sector*) shown at right. These integrals are called the *Fresnel integrals*. They play an important role in optical scattering.
(2 Marks)



Exercise 6.4. Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

by integrating the function $f(z) = (e^{iz} - 1)/(2iz)$ along the indented semi-circle shown at right.
(2 Marks)



Exercise 6.5. Let f be a holomorphic function on the disc D_0 centered at the origin and of radius R_0.

i) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\,\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi.$$

[Hint: Note that if $w = R^2/\overline{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity.]

ii) Show that

$$\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\varphi) + r^2} \qquad \text{for } z = re^{i\theta}.$$

(3 + 2 Marks)

Exercise 6.6. Let u be a real-valued function defined on the unit disc $D = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\triangle u(x, y) = 0 \qquad\qquad \text{for all } (x, y) \in D.$$

i) Deduce the *Poisson integral representation formula* from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) u(e^{i\varphi})\, d\varphi$$

where $P_r(\gamma)$ is the *Poison kernel* for the unit disc given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r \cos \varphi + r^2}.$$

ii) The *Dirichlet problem for the unit disc* is a boundary value problem for the Laplace equation, viz.

$$\begin{aligned} \triangle u(x, y) &= 0, & (x, y) &\in D,\\ u(x, y) &= f(x, y), & (x, y) &\in \partial D = S^1. \end{aligned}$$

where $f : S^1 \to \mathbb{R}$ is a continuous function. Prove that a solution to the Dirichlet problem for the unit disc is given by

$$u(x, y) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) f(\cos \varphi, \sin \varphi)\, d\varphi & r < 1 \\ f(\cos\theta, \sin\theta) & r = 1 \end{cases} \qquad (*)$$

whenever $x = r \cos \theta$, $y = r \sin \theta$. (You need to show the converse of your previous arguments, i.e., that u defined by $(*)$ is harmonic in D. It is clear from the definition that this u satisfies the boundary condition $u|_{\partial D} = f$ and is then the unique solution to the Dirichlet problem.)

(2 + 2 Marks)

Exercise 6.7. Use the Poisson formula for the Dirichlet problem in the unit disk to find the solution of the Dirichlet problem

$$\Delta_{r,\theta} u(r, \theta) = 0, \qquad (r, \theta) \in (0, 1) \times [-\pi, \pi), \qquad u(1, \varphi) = \begin{cases} -1 & -\pi \le \theta < 0, \\ 1 & 0 \le \theta < \pi, \end{cases}$$

in terms of elementary functions. Plot the graph of the solution using Mathematica. You may use that

$$\frac{1}{2\pi} \int \frac{1 - r^2}{1 + r^2 - 2r \cos(t - \varphi)}\, dt = \frac{1}{\pi} \arctan\Big(\frac{1 + r}{1 - r} \tan \frac{t - \varphi}{2}\Big).$$

(Pay careful attention the branches of the arctangent. The solution will be a continuous function of φ.)
(4 Marks)

Exercise 6.8. Suppose that $f : \Omega \to \mathbb{C}$ be holomorphic on a domain $\Omega \subset \mathbb{C}$. Let $z_0 \in \Omega$ be such that $B_r(z_0) \subset \Omega$ and suppose that for some $M > 0$,

$$|f(z)| \le M \qquad\qquad \text{for all } z \in B_r(z_0).$$

If $f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$ is the power series representation of f, show that

$$|c_n| \le \frac{M}{r^n}$$

(2 Marks)

Exercise 6.9.

i) Use Exercise 6.8 to prove *Liouville's Theorem*: Any bounded, entire function must be constant.

ii) Deduce the *Fundamental Theorem of Algebra*: Every polynomial of degree $n \ge 1$ has at least one zero. (*Instructions*: If f is a polynomial of degree $n \ge 1$, show that f is unbounded. Consider then $g = 1/f$ and show that g must be entire and bounded if f has no zero, contradicting Liouville's Theorem)

(2 + 2 Marks)

Exercise 6.10. Let $\Omega \subset \mathbb{C}$ be open and $g, h : \Omega \to \mathbb{C}$ be holomorphic at a point $z_0 \in \Omega$. Assume that h has a simple zero at z_0. Prove that

$$\operatorname{res}_{z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}.$$

(2 Marks)