Vv286 Honors Mathematics IV
Ordinary Differential Equations

Assignment 9

Date Due: 10:00 AM, Thursday, the 26th of November 2015

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Exercise 9.1. The equation

$$x'' - 2tx' + \lambda x = 0, \qquad \lambda \in \mathbb{R},$$

is known as the *Hermite differential equation*, and it appears in many areas of mathematics and physics.

i) Find two linearly independent solutions of the Hermite equation.

ii) Show that the Hermite equation has a polynomial solution of degree n if $\lambda = 2n$. (When normalized, this polynomial is the *Hermite polynomial* H_n encountered in Exercise 1 of Assignment 4. You do not need to prove this fact here.)

(3 + 2 Marks)

Exercise 9.2. Find two independent solutions for each of the following equations

$$\begin{aligned} 2ty'' + (1-2t)y' - y &= 0, \\ t^2y'' + (t-t^2)y' - y &= 0, \\ t^2y'' - t(1+t)y' + y &= 0. \end{aligned}$$

(3 + 3 + 4 Marks)

Exercise 9.3. The *Bessel equation of order* $\nu \geq 0$ is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

i) Use the Frobenius method to derive a solution of the Bessel equation. Show that one obtains a multiple of the *Bessel function of the first kind of order* $\nu \geq 0$,

$$J_\nu(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n+\nu)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{1}$$

Here Γ is the Euler gamma function.

ii) If 2ν is not an integer, find an independent second solution using the Frobenius method. Show that for $0 < \nu < 1$ it is a multiple of

$$J_{-\nu}(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+n-\nu)} \left(\frac{x}{2}\right)^{2n-\nu}. \tag{2}$$

What is the form of the second independent solution if $\nu > 1$, $2\nu \notin \mathbb{N}$?

iii) Show that $J_{-n} = (-1)^n J_n$ for $n \in \mathbb{N}$, so it is clear that $J_{-\nu}$ does not yield a second independent solution if $\nu \in \mathbb{N}$.

iv) Use the method of reduction of order to show that a second solution can be formally represented as

$$y_2(x) = J_\nu(x) \int \frac{dx}{x \cdot J_\nu^2(x)}$$

for any $\nu \geq 0$.

(3 + 3 + 1 + 3 Marks)

Exercise 9.4. The equation

$$y'' + xy = 0$$

is called *Airy's equation*. It was first discovered and analyzed by Airy in his study of optics.

i) Show that the substitution $u(t) = x^{-1/2}y(x)$ and $t = \frac{2}{3}x^{3/2}$ transforms Airy's equation into Bessel's equation of order $\nu = 1/3$.

ii) Use the above results to write down the general solution of Airy's equation first in terms of $J_{1/3}$ and $J_{-1/3}$ and then as series.

(3 + 2 Marks)