# Vv454 Partial Differential Equations and Boundary Value Problems

## Assignment 1

**Date Due: 8:00 AM, Friday, the 7$^{\text{th}}$ of March 2014**

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**Exercise 1.** Classify the following equations as linear, semi-linear, quasi-linear but not semi-linear, or fully non-linear:

i) $u_{xy} + u \cdot u_x = 0$, ii) $u_{xy} + u^2 \cdot u_x = 0$, iii) $u_{xy} + u \cdot u_x^2 = 0$, iv) $u_{xy}^2 + u \cdot u_x = 0$.

**(4 Marks)**

**Exercise 2.** The traffic equation can be written as

$$\rho_t + \frac{\partial}{\partial x} Q(\rho) = 0, \qquad (x,t) \in (a,b) \times (0,\infty),$$

where $Q(\varrho(x,t)) = \varrho(x,t)v(x,t)$. In this exercise, we will use the model

$$v(x,t) = \frac{v_{\max}}{\rho_{\max}} \big(\rho_{\max} - \rho(x,t)\big).$$

i) Suppose that $\rho(x,t) = \widetilde{\rho}(x/t)$. What ODE does $\widetilde{\rho}$ satisfy? Solve the ODE to obtain a particular solution

$$\rho(x,t) = \frac{\rho_{\max}}{2v_{\max}} \left( v_{\max} - \frac{x}{t} \right).$$

(This is known as a *similarity solution.*)

ii) Perform a change of variables to obtain an equation of the form

$$u_t + u \cdot u_x = 0, \qquad (x,t) \in (a,b) \times \mathbb{R}.$$

**(3 + 3 Marks)**

**Exercise 3.** An insect's digestive tract[1] is modelled as a tube of length $l$ and constant cross-sectional area $A$. Nutrients of concentration $u(x,t)$ flow through the tract with constant speed $c$ and are absorbed at a rate proportional to $\sqrt{u}$. The tract is empty at $t = 0$ and nutrients are introduced at the mouth (at $x = 0$) with a constant concentration $u_0$ for $t > 0$. Derive the PDE for $u$ with suitable boundary and initial conditions, and find the solution.
**(2 + 2 Marks)**

**Exercise 4.** Suppose that a motionless liquid is contained in a thin, infinitely long tube or pipe and that a chemical substance, e.g., a dye, is diffusing through the liquid. The concentration of the dye can be described by a function $u\colon \mathbb{R} \to \mathbb{R}$, where $u(x)$ is the concentration at position $x$. In any section $[a,b]$ of the pipe, the change in amount of dye is given by the difference in dye diffusing out of and diffusing into the section. Assume that the dye diffuses according to *Fick's law*: the flow speed $v$ of the dye is proportional to the gradient of the the concentration of the dye, $v = -ku_x$.

i) Suppose that $k$ is constant everywhere. Derive the PDE

$$u_t = k u_{xx}, \qquad (x,t) \in \mathbb{R}^2,$$

for the concentration $u(x,t)$.

ii) Suppose the liquid is flowing in positive $x$-direction at a constant velocity $V > 0$. Derive the PDE describing $u(x,t)$.

(This equation also describes heat conduction, brownian motion and many other phenomena.)
**(4 + 3 Marks)**

[1] Exercise adapted from R. Piché, Exercises 1 of the course "Partial Differential Equations", Spring 2011, Tampere University of Technology.

**Exercise 5.** Use the divergence theorem (Gauß's theorem) to prove *Green's first identity*:

Let $\Omega \subset \mathbb{R}^3$ be a sufficiently nice open set, $\varphi \in C^2(\Omega)$, $\psi \in C^1(\Omega)$. Then

$$\int_\Omega \big(\psi \Delta \varphi + \langle \nabla \varphi, \nabla \psi \rangle\big)\, dx = \int_{\partial \Omega} \psi \cdot \frac{\partial \varphi}{\partial n}\, dA,$$

where the surface $\partial\Omega$ has positive orientation.

Next, prove *Green's second identity*:

Let $\Omega \subset \mathbb{R}^3$ be a sufficiently nice open set, $\varphi, \psi \in C^2(\Omega)$. Then

$$\int_\Omega \big(\psi \Delta \varphi - \varphi \Delta \psi\big)\, dx = \int_{\partial \Omega} \Big(\psi \cdot \frac{\partial \varphi}{\partial n} - \varphi \cdot \frac{\partial \psi}{\partial n}\Big)\, dA,$$

where the surface $\partial\Omega$ has positive orientation.

**(3 + 3 Marks)**

**Exercise 6.** Find the general solution for the following PDEs (explicitly if possible, otherwise as an implicit equation), where $(x, y) \in \mathbb{R}^2$:

i) $(x + 2y)u_x - yu_y = 0,$

ii) $(y + u)u_x - (x + u)u_y + x - y = 0.$

**(2 + 2 Marks)**

**Exercise 7.** Solve the following Cauchy problems in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively:

i) $u_x + xu_y = u^2, \quad u(x, 0) = e^x,$

ii) $y^{-1}u_x + u_y = 0, \quad u(x, 1) = x^2,$

iii) $u_x + xu_y - u_z = u, \quad u(x, y, 1) = x + y.$

**(3 × 2 Marks)**

**Exercise 8.** Consider the following Cauchy problem

$$u_t - t^2 u_x = -u, \qquad u(x, 0) = x^2, \qquad (x, t) \in \mathbb{R} \times \mathbb{R}_+.$$

i) Sketch (or plot using a computer) the characteristics for the Cauchy problem in the $x$-$t$ plane.

ii) Transform the PDE into an ODE, solve the ODE and transform back to find the general solution of the PDE. Then find the solution of the Cauchy problem.

**(1 + 2 Marks)**