Vv454 Partial Differential Equations and Boundary Value Problems

Assignment 3

Date Due: 8:00 AM, Friday, the 21st of March 2014

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Exercise 1. Reducing a PDE to its normal form can be useful for finding general solutions. The normal form of a hyperbolic equation is

$$v_{\xi\eta}(\xi,\eta) = 0.$$

Let us take this equation on the domain $\mathbb{R}^2 \ni (\xi,\eta)$. Show, by integrating twice, that the solution on this domain is given by

$$v(\xi,\eta) = f(\xi) + g(\eta)$$

where $f, g\colon \mathbb{R} \to \mathbb{R}$ are arbitrary functions (twice continuously differentiable if the solution is to be classical).
(2 Marks)

Exercise 2. Reduce the equation

$$u_{xx} + 5u_{xy} + 6u_{yy} = 0$$

to normal form, integrate the normal form to find the general solution, and then give general solutions to the original equation.
(4 Marks)

Exercise 3. Consider the equation[1]

$$4y^2u_{xx} + 2(1-y^2)u_{xy} - u_{yy} - \frac{2y}{1+y^2}(2u_x - u_y) = 0.$$

i) Find the normal form of the equation.

ii) Find the general solution of the equation.

iii) Find the solution $u(x,y)$ which satisfies

$$u(x,0) = g(x), \qquad u_y(x,0) = f(x), \qquad f,g \in C^2(\mathbb{R}).$$

(2 + 2 + 2 Marks)

Exercise 4. Consider the equation[2]

$$u_{xx} + (1+y^2)^2u_{yy} - 2y(1+y^2)u_y = 0.$$

i) Find the normal form of the equation.

ii) Find the general solution of the equation.

iii) Find the solution $u(x,y)$ which satisfies

$$u(x,0) = g(x), \qquad u_y(x,0) = f(x), \qquad f,g \in C^2(\mathbb{R}).$$

(2 + 2 + 2 Marks)

[1] See *Pinchover/Rubinstein*, Exercise 3.8
[2] See *Pinchover/Rubinstein*, Exercise 3.6

Exercise 5. Non-dimensionalize the cable equation derived in the lecture,

$$u_t = D \cdot u_{xx} - \beta u.$$

(2 Marks)

Exercise 6. Consider the equation

$$au_{xx} - bu_t + cu = 0, \qquad (x,t) \in \mathbb{R}^2,$$

where $a, b, c \in \mathbb{R}$, $b \neq 0$ are constants.

i) Fix any $\delta \in \mathbb{R}$ and suppose that $u(x,t) := e^{\delta t} w(x,t)$ satisfies the PDE. Find the PDE that must be satisfied by w.

ii) Show that with given $a, b, c \in \mathbb{R}$, $b \neq 0$, the constant δ can be chosen so that the PDE for w is just the $(1+1)$-dimensional heat equation.

This shows that solutions to the cable equation can be found by solving the heat equation.
(3 + 1 Marks)

Exercise 7. In the lecture, we derived the formula

$$\Delta_{(\xi,\eta,\zeta)} = \frac{1}{abc}\left(\frac{\partial}{\partial\xi}\left(\frac{bc}{a}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{ac}{b}\frac{\partial}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{ab}{c}\frac{\partial}{\partial\zeta}\right)\right)$$

for the Laplacian in orthogonal curvilinear coordinates (ξ, η, ζ) in $\mathbb{R}^3$, where a, b, c are the moduli of the tangent vectors of the ξ-, η- and ζ-lines, respectively.

i) Using the same method of proof, find an analogous formula for the Laplacian in orthogonal curvilinear coordinates (ξ, η) in $\mathbb{R}^2$. (The Divergence Theorem may, for example, be replaced by Green's Theorem.)

ii) Use the formula obtained in i) to write down the Laplacian $\Delta_{(r,\varphi)}$ for coordinates defined by $x = r\cos\varphi$, $y = r\sin\varphi$, $r \in \mathbb{R}_+$, $\varphi \in [0, 2\pi)$.

(3 + 1 Marks)

Exercise 8. The electrostatic problem of two equal and opposite point charges in $\mathbb{R}^2$, located at $P_1 = (-a, 0)$ and $P_2 = (a, 0)$ for some $a > 0$, can be treated by introducing *bipolar polar coordinates* (σ, τ) defined through

$$x = a\frac{\sinh\tau}{\cosh\tau - \cos\sigma}, \qquad y = a\frac{\sin\sigma}{\cosh\tau - \cos\sigma},$$

where $\tau \in \mathbb{R}$ and $\sigma \in [-\pi, \pi]$.

i) Create a sketch (perhaps using Mathematica) showing several σ-curves (along which τ is constant).

ii) Create a sketch (perhaps using Mathematica) showing several τ-curves (along which σ is constant).

iii) Show that σ is the (signed) angle P_1PP_2.

iv) Show that $\tau = \ln(d_1/d_2)$, where $d_1 = |P - P_1|$ and $d_2 = |P - P_2|$.

v) Use the formula derived in Exercise 7, part i), to express the Laplacian in the coordinates (σ, τ). (If you have not solved this exercise, use the chain rule instead.)

vi) For the problem in three dimensions, one introduces *bispherical coordinates* (σ, τ, ϕ) through

$$x = a\frac{\sin\sigma}{\cosh\tau - \cos\sigma}\cos\phi, \qquad y = a\frac{\sin\sigma}{\cosh\tau - \cos\sigma}\sin\phi, \qquad z = a\frac{\sinh\tau}{\cosh\tau - \cos\sigma},$$

where $\tau \in \mathbb{R}$ and $\sigma \in [-\pi, \pi]$, $\phi \in [0, 2\pi)$. Use a computer to plot the three surfaces in $\mathbb{R}^3$ where

(a) $\sigma = \pi/5$,
(b) $\tau = 0.4$,
(c) $\phi = \pi/4$.

vii) Express the Laplace operator in terms of bispherical coordinates.

(1 + 1 + 2 + 2 + 2 + 2 + 2 Marks)