Vv454 Partial Differential Equations and Boundary Value Problems

Assignment 4

Date Due: 8:00 AM, Friday, the 28th of March 2014

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Exercise 1. Solve the problem

$$u_{tt} - 4u_{xx} = e^x + \sin t, \qquad u(x,0) = 0, \qquad u_t(x,0) = \frac{1}{1+x^2}, \qquad x \in \mathbb{R},\ t > 0.$$

(3 Marks)

Exercise 2. Solve the problem

$$u_{tt} - 4u_{xx} = 6t, \qquad u(x,0) = x, \qquad u_t(x,0) = 0, \qquad x \in \mathbb{R},\ t > 0.$$

(3 Marks)

Exercise 3. A pressure wave generated as a result of an explosion satisfies

$$P_{tt} - 16P_{xx} = 0, \qquad x \in \mathbb{R},\ t > 0,$$

where $P(x,t)$ is the pressure at the point x and time t. The initial conditions at the explosion time t are

$$P(x,0) = \begin{cases} 10 & |x| < 1, \\ 0 & \text{otherwise,} \end{cases} \qquad P_t(x,0) = \begin{cases} 1 & |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

A building is located at the point $x_0 = 10$. The engineer who designed the building determined that it will sustain a pressure up to $P = 6$. Find the time t_0 when the pressure at the building is maximal. Will the building collapse?
(4 Marks)

Exercise 4. Vibrations of small amplitude of a homogeneous beam of length l can be described by the *beam equation*

$$u_{tt} - \gamma^2 u_{xxxx} = 0, \qquad (x,t) \in (0,l) \times \mathbb{R}_+, \tag{1}$$

where $\gamma > 0$ is a constant. Suppose that the beam is clamped at both ends, i.e.,

$$u(0,t) = u_x(0,t) = u(l,t) = u_x(l,t) = 0. \tag{2}$$

Show that the energy

$$E(t) := \frac{1}{2}\int_0^l \left(|u_t(x,t)|^2 + \gamma^2 |u_{xx}(x,t)|^2\right) dx, \qquad t \geq 0,$$

is conserved, i.e., $E(t)$ is actually constant. Use this to prove that there is at most one solution to (1) with boundary conditions (2) and initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad x \in [0,l].$$

(4 Marks)

Exercise 5. Orthonormalize the monomials $\{1, x, x^2, x^3, x^4\}$ in the space $L^2(\mathbb{R}, e^{-x^2}\,dx)$ using the Gram-Schmidt procedure. The polynomials obtained in this way are called the *Hermite polynomials*.
(2 Marks)

Exercise 6. Calculate the Fourier series of the function f defined on $[-1, 1]$ and given by $f(x) = x^2$. Evaluate the series at a suitable point to find the value of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

By evaluating the series at a different point, find the value of

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}.$$

(4 Marks)