Vv454 Partial Differential Equations and Boundary Value Problems

Assignment 6

Date Due: 8:00 AM, Friday, the 11^{th} of April 2014

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Exercise 1.

i) The ends of a copper rod with insulated mantle and 100 cm long are maintained at $0°$ C. Suppose that the center of the bar is heated to $100°$ C by an external heat source and that this situation is maintained until a steady-state results. Find this steady-state temperature distribution.

ii) At a time $t = 0$ (after the steady-state of part i) has been reached) the heat source is removed. At the same instant the end $x = 0$ is placed in thermal contact with a reservoir at $20°$ C while the other end remains at $0°$ C. Find the temperature $u(x, t)$ as a function of position and time. Create a three-dimensional plot of the graph of u using Mathematica.

iii) What limiting value does the temperature at the center of the rod approach after a long time? How much time must elapse before the center of the rod cools to within 1 degree of its limiting value?

(3 × 2 Marks)

Exercise 2. Consider the equation for a vibrating beam of length $l > 0$,

$$u_{tt} + c^2 u_{xxxx} = 0, \qquad (x, t) \in (0, l) \times \mathbb{R}_+.$$

Find the solution to the initial-boundary value problem

$$u(0, t) = u(l, t) = 0, \qquad u_{xx}(0, t) = u_{xx}(l, t) = 0, \qquad t \in \mathbb{R}_+,$$
$$u(x, 0) = x(l - x), \qquad u_t(x, 0) = 0, \qquad x \in (0, l).$$

(4 Marks)

Exercise 3. Use a separation-of-variables approach to solve the damped wave equation

$$c^2 u_{xx} - u_{tt} - \mu u_t = 0, \qquad (x, t) \in (0, L) \times \mathbb{R}_+,\ L > 0,$$

with Dirichlet boundary conditions

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad t > 0$$

and initial conditions

$$u(x, 0) = \sin\left(\frac{\pi x}{L}\right), \qquad u_t(x, 0) = 0, \qquad x \in [0, L].$$

(4 Marks)

Exercise 4. Solve the equation

$$u_{xx} + u_{yy} = u, \qquad (x, y) \in (0, \pi) \times (0, a), \quad a > 0,$$

with boundary conditions

$$u(0, y) = u(\pi, y) = u(x, 0) = 0, \qquad u(x, a) = 1, \qquad (x, y) \in [0, \pi] \times [0, a].$$

(4 Marks)

Exercise 5. Solve the heat equation

$$u_{xx} - u_t = 0, \qquad (x,t) \in (0, 1/2) \times \mathbb{R}_+$$

with Neumann boundary conditions

$$u_x(0,t) = 0, \qquad u_x(1/2, t) = 1, \qquad t > 0$$

and initial temperature distribution

$$u(x, 0) = 0, \qquad x \in [0, 1/2].$$

(4 Marks)

Exercise 6. Solve the heat equation

$$\alpha^2 u_{xx} - u_t = 0, \qquad (x,t) \in (0,1) \times \mathbb{R}_+$$

with Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 1, \qquad t > 0$$

and initial temperature distribution

$$u(x,0) = 0, \qquad x \in [0,1].$$

(4 Marks)

Exercise 7. Solve the inhomogeneous heat equation

$$u_{xx} - u_t = -2x, \qquad (x,t) \in (0,1) \times \mathbb{R}_+$$

with Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t > 0$$

and initial temperature distribution

$$u(x,0) = x - x^2, \qquad x \in [0,1].$$

(4 Marks)

Exercise 8. Show that the telegraph equation with $\alpha, \beta > 0$,

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}, \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

with the condition

$$\sup_{x \in \mathbb{R}_+} |u(x,t)| < \infty, \qquad t \in \mathbb{R}_+$$

and initial signal

$$u(0,t) = U_0 \cos(\omega t), \qquad \omega, U_0 > 0,$$

does not have a solution of the form $u(x,t) = X(x) \cdot T(t)$. Next, show that there exists a solution of the from

$$u(x,t) = U_0 e^{-Ax} \cos(\omega t + Bx)$$

for certain constants A and B. How are A and B determined from α, β, ω? (Thus, not every problem has a solution that can be found from a separation-of-variables approach.)
(2 + 3 Marks)

Exercise 9. Use the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} e^{-x^2/2} \, dx = e^{-\xi^2/2}$$

to prove that

$$\int_0^{\infty} \cos\big(\omega(x-y)\big) e^{-c^2\omega^2 t} \, d\omega = \frac{\sqrt{\pi}}{2} \frac{1}{c\sqrt{t}} e^{-\frac{(x-y)^2}{4c^2 t}}.$$

where $x, y \in \mathbb{R}$ and $c, t > 0$.
(2 Marks)

Exercise 10. Use the function

$$b(x) = \begin{cases} e^{-\frac{1}{x^2-1}} & -1 \leq x \leq 1, \\ 0 & |x| > 1. \end{cases}$$

to construct a function $\chi \in C^{\infty}(\mathbb{R})$ such that

$$\chi(x) = \begin{cases} 0 & x < -\varepsilon, \\ 1 & x > \varepsilon \end{cases}$$

and χ takes on suitable values in the interval $[-\varepsilon, \varepsilon]$. Use this function and Mathematica (or Matlab or similar software) to plot a smooth function $f \in C^{\infty}(\mathbb{R})$ such that

$$f(x) = \begin{cases} e^x & |x| < 1, \\ 0 & |x| > 101/100. \end{cases}$$

(Submit the plot as well as the source code for this exercise.)
(3 Marks)